Lecture 25

1. Expectations, continued

Theorem 25.1. *If X is a positive random variable with density f, then*

$$\mathrm{E}(X) = \int_0^\infty \mathrm{P}\{X > x\}\, dx = \int_0^\infty (1 - F(x))\, dx.$$

Proof. The second identity is a consequence of the fact that $1 - F(x) = \mathrm{P}\{X > x\}$. In order to prove the first identity note that $\mathrm{P}\{X > x\} = \int_x^\infty f(y)\, dy$. Therefore,

$$\begin{aligned}\int_0^\infty \mathrm{P}\{X > x\}\, dx &= \int_0^\infty \int_x^\infty f(y)\, dy\, dx \\ &= \int_0^\infty f(y) \int_0^y dx\, dy \\ &= \int_0^\infty y f(y)\, dy,\end{aligned}$$

and this is $\mathrm{E}(X)$. □

Question: Why do we need X to be positive? [To find the answer you need to think hard about the change of variables formula of calculus.]

Theorem 25.2. *If $\int_{-\infty}^\infty |g(a)| f(a)|\, da < \infty$, then*

$$\mathrm{E}[g(X)] = \int_{-\infty}^\infty g(a) f(a)\, da.$$

Proof. I will prove the result in the special case that $g(x) \geq 0$, but will not assume that $\int_{-\infty}^\infty g(a) f(a)\, da < \infty$.

The preceding theorem implies that

$$\mathrm{E}[g(X)] = \int_0^\infty \mathrm{P}\{g(X) > x\}\, dx.$$

But $\mathrm{P}\{g(X) > x\} = \mathrm{P}\{X \in A\}$ where $A = \{y : \ g(y) > x\}$. Therefore,

$$\begin{aligned}\mathrm{E}[g(X)] &= \int_0^\infty \int_{\{y:\ g(y)>x\}} f(y)\, dy\, dx \\ &= \int_0^\infty \int_0^{g(y)} f(y)\, dx\, dy \\ &= \int_0^\infty g(y) f(y)\, dy,\end{aligned}$$

as needed. □

Properties of expectations:

(1) If $g(X)$ and $h(X)$ have finite expectations, then

$$\mathrm{E}\left[g(X) + h(X)\right] = \mathrm{E}[g(X)] + \mathrm{E}[h(X)].$$

(2) If $\mathrm{P}\{a \le X \le b\} = 1$ then $a \le \mathrm{E}X \le b$.

(3) Markov's inequality: If $h(x) \ge 0$, then

$$\mathrm{P}\left\{h(X) \ge a\right\} \le \frac{\mathrm{E}[h(X)]}{a} \qquad \text{for all } a > 0.$$

(4) Cauchy–Schwarz inequality:

$$\mathrm{E}[X^2] \ge \{\mathrm{E}(|X|)\}^2.$$

In particular, if $\mathrm{E}[X^2] < \infty$, then $\mathrm{E}(|X|)$ and $\mathrm{E}X$ are both finite.

Definition 25.3. The *variance* of X is defined as

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - |\mathrm{E}X|^2.$$

Alternative formula:

$$\mathrm{Var}(X) = \mathrm{E}\left[(X - \mathrm{E}X)^2\right].$$

Example 25.4 (Moments of Uniform$(0\,,1)$)**.** If X is uniform$(0\,,1)$, then for all integers $n \ge 1$,

$$\mathrm{E}(X^n) = \int_0^1 x^n\, dx = \frac{1}{n+1}.$$

Example 25.5 (Moments of $N(0\,,1)$)**.** Compute $\mathrm{E}(X^n)$, where $X = N(0\,,1)$ and $n \geq 1$ is an integer:

$$\begin{aligned}\mathrm{E}(X^n) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a^n e^{-a^2/2}\, da \\ &= 0 \quad \text{if } n \text{ is odd, by symmetry.}\end{aligned}$$

If n is even, then

$$\begin{aligned}\mathrm{E}(X^n) &= \frac{2}{\sqrt{2\pi}} \int_0^{\infty} a^n e^{-a^2/2}\, da = \sqrt{\frac{2}{\pi}} \int_0^{\infty} a^n e^{-a^2/2}\, da \\ &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} (2z)^{n/2} e^{-z} \underbrace{\left((2z)^{-1/2}\, dz\right)}_{da} \qquad \left(z = a^2/2 \ \Leftrightarrow\ a = \sqrt{2z}\right) \\ &= \frac{2^{n/2}}{\sqrt{\pi}} \int_0^{\infty} z^{(n-1)/2} e^{-z}\, dz \\ &= \frac{2^{n/2}}{\sqrt{\pi}} \Gamma\left(\frac{n+1}{2}\right).\end{aligned}$$