1. Functions of a random variable, continued

Example 24.1. It is best to try to work on these problems on a case-by-case basis. Here is an example where you need to do that. Consider X to be a uniform $(0\,,1)$ random variable, and define $Y = \sin(\pi X/2)$. Because $X \in (0\,,1)$, it follows that $Y \in (0\,,1)$ as well. Therefore, $F_Y(a) = 0$ if $a < 0$, and $F_Y(1) = 1$ if $a > 1$. If $0 \le a \le 1$, then

$$F_Y(a) = \mathrm{P}\left\{\sin\left(\frac{\pi X}{2}\right) \le a\right\} = \mathrm{P}\left\{X \le \frac{2}{\pi}\arcsin a\right\} = \frac{2}{\pi}\arcsin a.$$

You need to carefully plot the arcsin curve to deduce this. Therefore,

$$f_Y(a) = \begin{cases} \dfrac{2}{\pi\sqrt{1-a^2}} & \text{if } 0 < a < 1,\\ 0 & \text{otherwise.}\end{cases}$$

Finally, a transformation of a continuous random variable into a discrete one ....

Example 24.2. Suppose X is uniform $(0\,,1)$ and define $Y = \lfloor 2X \rfloor$ to be the largest integer $\le 2X$. Find f_Y.

First of all, we note that Y is discrete. Its possible values are 0 (this is when $0 < X < 1/2$) and 1 (this is when $1/2 < X < 1$). Therefore,

$$f_Y(0) = \mathrm{P}\left\{0 < X < \frac{1}{2}\right\} = \int_0^{1/2} dy = \frac{1}{2} = 1 - f_Y(1) = \frac{1}{2}.$$

This is thrown in just so we remember that it is entirely possible to start out with a continuous random variable, and then transform it into a discrete one.

2. Expectation

If X is a continuous random variable with density f, then its *expectation* is defined to be

$$\mathrm{E}(X) = \int_{-\infty}^{\infty} x f(x)\, dx,$$

provided that either $X \geq 0$, or $\int_{-\infty}^{\infty} |x| f(x)\, dx < \infty$.

Example 24.3 (Uniform)**.** Suppose X is uniform $(a\,, b)$. Then,

$$\mathrm{E}(X) = \int_a^b x \frac{1}{b-a}\, dx = \frac{1}{2}\frac{b^2 - a^2}{b-a}.$$

It is easy to check that $b^2 - a^2 = (b-a)(b+a)$, whence

$$\mathrm{E}(X) = \frac{b+a}{2}.$$

N.B.: The formula of the first example on page 303 of your text is wrong.

Example 24.4 (Gamma)**.** If X is Gamma$(\alpha\,, \lambda)$, then for all positive values of x we have $f(x) = \lambda^\alpha/\Gamma(\alpha) x^{\alpha-1} e^{-\lambda x}$, and $f(x) = 0$ for $x < 0$. Therefore,

$$\begin{aligned}
\mathrm{E}(X) &= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^\alpha e^{-\lambda x}\, dx \\
&= \frac{1}{\lambda\Gamma(\alpha)} \int_0^\infty z^\alpha e^{-z}\, dz \qquad (z = \lambda x) \\
&= \frac{\Gamma(\alpha+1)}{\lambda\Gamma(\alpha)} \\
&= \frac{\alpha}{\lambda}.
\end{aligned}$$

In the special case that $\alpha = 1$, this is the expectation of an exponential random variable with parameter λ.

Example 24.5 (Normal)**.** Suppose $X = N(\mu\,, \sigma^2)$. That is,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2} \right).$$

Then,

$$\begin{aligned}
\mathrm{E}(X) &= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} x\exp\left(-\frac{(x-\mu)^2}{2}\right)\,dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(\mu+\sigma z)e^{-z^2/2}\,dz \qquad (z=(x-\mu)/\sigma) \\
&= \mu\underbrace{\int_{-\infty}^{\infty}\frac{e^{-z^2/2}}{\sqrt{2\pi}}\,dz}_{1}+\frac{\sigma}{\sqrt{2\pi}}\underbrace{\int_{-\infty}^{\infty} ze^{-z^2/2}\,dz}_{\text{0, by symmetry}} \\
&= \mu.
\end{aligned}$$

Example 24.6 (Cauchy)**.** In this example, $f(x) = \pi^{-1}(1+x^2)^{-1}$. Note that the expectation is defined only if the following limit exists regardless of how we let n and m tend to ∞:

$$\int_{-m}^{n}\frac{y}{1+y^2}\,dy.$$

Now I argue that the limit does not exist; I do so by showing two different choices of $(n\,,m)$ which give rise to different limiting "integrals."

First suppose $m = n$, so that by symmetry,

$$\int_{-n}^{n}\frac{y}{1+y^2}\,dy = 0.$$

Let $n \to \infty$ to obtain zero as the limit of the left-hand side.

Next, suppose $m = 2n$. Again by symmetry,

$$\begin{aligned}
\int_{-2n}^{n}\frac{y}{1+y^2}\,dy &= \int_{-2n}^{-n}\frac{y}{1+y^2}\,dy \\
&= -\int_{n}^{2n}\frac{y}{1+y^2}\,dy \\
&= -\frac{1}{2}\int_{1+n^2}^{1+4n^2}\frac{dz}{z} \qquad (z=1+y^2) \\
&= -\frac{1}{2}\ln\left(\frac{1+4n^2}{1+n^2}\right) \\
&\to -\frac{1}{2}\ln 4 \quad \text{as } n\to\infty.
\end{aligned}$$

Therefore, the Cauchy density does not have a well-defined expectation. [That is not to say that the expectation is well defined, but infinite.]