1. Functions of a continuous random variable, continued

The problem: $Y = g(X)$; find f_Y in terms of f_X.

The solution: First compute F_Y, by hand, in terms of F_X, and then use the fact that $F_Y' = f_Y$ and $F_X' = f_X$.

Example 23.1. Suppose X has density f_X. Then let us find the density function of $Y = X^2$. Again, we seek to first compute F_Y. Now, for all $a > 0$,

$$F_Y(a) = \mathrm{P}\{X^2 \le a\} = \mathrm{P}\left\{-\sqrt{a} \le X \le \sqrt{a}\right\} = F_X\left(\sqrt{a}\right) - F_X\left(-\sqrt{a}\right).$$

Differentiate $[d/da]$ to find that

$$f_Y(a) = \frac{f_X\left(\sqrt{a}\right) + f_X\left(-\sqrt{a}\right)}{2\sqrt{a}}$$

On the other hand, $f_Y(a) = 0$ if $a \le 0$. For example, consider the case that X is standard normal. Then,

$$f_{X^2}(a) = \begin{cases} \dfrac{e^{-a}}{\sqrt{2\pi a}} & \text{if } a > 0, \\ 0 & \text{if } a \le 0. \end{cases}$$

Or if X is Cauchy, then

$$f_{X^2}(a) = \begin{cases} \dfrac{1}{\pi\sqrt{a}(1+a)} & \text{if } a > 0, \\ 0 & \text{if } a \le 0. \end{cases}$$

Or if X is uniform $(0\,,1)$, then

$$f_{X^2}(a) = \begin{cases} \dfrac{1}{2\sqrt{a}} & \text{if } 0 < a < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Example 23.2. Suppose $\mu \in \mathbf{R}$ and $\sigma > 0$ are fixed constants, and define $Y = \mu + \sigma X$. Find the density of Y in terms of that of X. Once again,

$$F_Y(a) = \mathrm{P}\left\{\mu + \sigma X \leq a\right\} = \mathrm{P}\left\{X \leq \frac{a-\mu}{\sigma}\right\} = F_X\left(\frac{a-\mu}{\sigma}\right).$$

Therefore,

$$f_Y(a) = \frac{1}{\sigma} f_X\left(\frac{a-\mu}{\sigma}\right).$$

For example, if X is standard normal, then

$$f_{\mu+\sigma X}(a) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

This is the socalled $N(\mu\,,\sigma^2)$ density.

Example 23.3. Suppose X is uniformly distributed on $(0\,,1)$, and define

$$Y = \begin{cases} 0 & \text{if } 0 \leq X < \frac{1}{3}, \\ 1 & \text{if } \frac{1}{3} \leq X < \frac{2}{3}, \\ 2 & \text{if } \frac{2}{3} \leq X < 1. \end{cases}$$

Then, Y is a discrete random variable with mass function,

$$f_Y(x) = \begin{cases} \frac{1}{3} & \text{if } x = 0,\ 1, \text{ or } 2, \\ 0 & \text{otherwise.} \end{cases}$$

For instance, in order to compute $f_Y(1)$ we note that

$$f_Y(1) = \mathrm{P}\left\{\frac{1}{3} \leq X < \frac{2}{3}\right\} = \int_{1/3}^{2/3} \underbrace{f_X(y)}_{\equiv 1}\, dy = \frac{1}{3}.$$

Example 23.4. Another common transformation is $g(x) = |x|$. In this case, let $Y = |X|$ and note that if $a > 0$, then

$$F_Y(a) = \mathrm{P}\{-a < X < a\} = F_X(a) - F_X(-a).$$

Else, $F_Y(a) = 0$. Therefore,

$$f_Y(a) = \begin{cases} f_X(a) + f_X(-a) & \text{if } a > 0, \\ 0 & \text{if } a \leq 0. \end{cases}$$

For instance, if X is standard normal, then

$$f_{|X|}(a) = \begin{cases} \frac{2}{\pi}\, e^{-a^2/2} & \text{if } a > 0, \\ 0 & \text{if } a \leq 0. \end{cases}$$

Or if X is Cauchy, then

$$f_{|X|}(a) = \begin{cases} \frac{2}{\pi}\, \frac{1}{1+a^2} & \text{if } a > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Can you guess $f_{|X|}$ when X is uniform $(-1\,,1)$?

Example 23.5. It is best to try to work on these problems on a case-by-case basis. Here is an example where you need to do that. Consider X to be a uniform $(0\,,1)$ random variable, and define $Y = \sin(\pi X/2)$. Because $X \in (0\,,1)$, it follows that $Y \in (0\,,1)$ as well. Therefore, $F_Y(a) = 0$ if $a < 0$, and $F_Y(1) = 1$ if $a > 1$. If $0 \leq a \leq 1$, then

$$F_Y(a) = \mathrm{P}\left\{\sin\left(\frac{\pi X}{2}\right) \leq a\right\} = \mathrm{P}\left\{X \leq \frac{2}{\pi}\arcsin a\right\} = \frac{2}{\pi}\arcsin a.$$

You need to carefully plot the arcsin curve to deduce this. Therefore,

$$f_Y(a) = \begin{cases} \dfrac{2}{\pi\sqrt{1-a^2}} & \text{if } 0 < a < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, a transformation of a continuous random variable into a discrete one ....

Example 23.6. Suppose X is uniform $(0\,,1)$ and define $Y = \lfloor 2X \rfloor$ to be the largest integer $\leq 2X$. Find f_Y.

First of all, we note that Y is discrete. Its possible values are 0 (this is when $0 < X < 1/2$) and 1 (this is when $1/2 < X < 1$). Therefore,

$$f_Y(0) = \mathrm{P}\left\{0 < X < \frac{1}{2}\right\} = \int_0^{1/2} dy = \frac{1}{2} = 1 - f_Y(1) = \frac{1}{2}.$$