Lecture 22

1. Examples of continuous random variables

Example 22.1 (Standard normal density)**.** I claim that

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$$

defines a density function. Clearly, $\phi(x) \geq 0$ and is continuous at all points x. So it suffices to show that the area under ϕ is one. Define

$$A = \int_{-\infty}^{\infty} \phi(x)\, dx.$$

Then,

$$\begin{aligned}
A^2 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2+y^2}{2}\right) dx\, dy \\
&= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} \exp\left(-\frac{r^2}{2}\right) r\, dr\, d\theta.
\end{aligned}$$

Let $s = r^2/2$ to find that the inner integral is $\int_0^\infty \exp(-s)\, ds = 1$. Therefore, $A^2 = 1$ and hence $A = 1$, as desired. [Why is A not -1?]

The distribution function of ϕ is

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2/2}\, dz.$$

One can prove that there is "no nice formula" that "describes" $\Phi(x)$ for all x (theorem of Liouville). Usually, people use tables of integrals to evaluate $\Phi(x)$ for concrete values of x.

Example 22.2 (Gamma densities)**.** Choose and fix two numbers (parameters) $\alpha, \lambda > 0$. The *gamma density* with parameters α and λ is the probability density function that is proportional to

$$\begin{cases} x^{\alpha-1}e^{-\lambda x} & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Now,

$$\int_0^\infty x^{\alpha-1}e^{-\lambda x}\,dx = \frac{1}{\lambda^\alpha}\int_0^\infty y^{\alpha-1}e^{-y}\,dy.$$

Define the *gamma function* as

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1}e^{-y}\,dy \qquad \text{for all } \alpha > 0.$$

One can prove that there is "no nice formula" that "describes" $\Gamma(\alpha)$ for all α (theorem of Liouville). Thus, the best we can do is to say that the following is a Gamma density with parameters $\alpha, \lambda > 0$:

$$f(x) = \begin{cases} \dfrac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1}e^{-\lambda x} & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

You can probably guess by now (and correctly!) that $F(x) = \int_{-\infty}^x f(y)\,dy$ cannot be described by nice functions either. Nonetheless, let us finish by making the observation that $\Gamma(\alpha)$ is computable for some reasonable values of $\alpha > 0$. The key to unraveling this remark is the following "reproducing property":

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha) \qquad \text{for all } \alpha > 0. \tag{18}$$

The proof uses integration by parts:

$$\begin{aligned} \Gamma(\alpha+1) &= \int_0^\infty x^\alpha e^{-x}\,dx \\ &= \int_0^\infty u(x)v'(x)\,dx, \end{aligned}$$

where $u(x) = x^\alpha$ and $v'(x) = e^{-x}$. Integration by parts states that[1]

$$\int uv' = uv - \int v'u \qquad \text{for indefinite integrals.}$$

[1]This follows immediately from integrating the product rule: $(uv)' = u'v + uv'$.

Evidently, $u'(x) = \alpha x^{\alpha-1}$ and $v(x) = -e^{-x}$. Hence,

$$\begin{aligned}\Gamma(\alpha+1) &= \int_0^\infty x^\alpha e^{-x}\,dx \\ &= uv\Big|_0^\infty - \int_0^\infty v'u \\ &= \left(-\alpha x^{\alpha-1}e^{-x}\right)\Big|_0^\infty + \alpha\int_0^\infty x^{\alpha-1}e^{-x}\,dx.\end{aligned}$$

The first term is zero, and the second (the integral) is $\alpha\Gamma(\alpha)$, as claimed. Now, it easy to see that $\Gamma(1) = \int_0^\infty e^{-x}\,dx = 1$. Therefore, $\Gamma(2) = 1\times\Gamma(1) = 1$, $\Gamma(3) = 2\times\Gamma(2) = 2$, $\ldots$, and in general,

$$\Gamma(n+1) = n! \qquad \text{for all integers } n \geq 0.$$

2. Functions of a continuous random variable

The basic problem: If $Y = g(X)$, then how can we compute f_Y in terms of f_X?

Example 22.3. Suppose X is uniform on $(0\,,1)$, and $Y = -\ln X$. Then, we compute f_Y by first computing F_Y, and then using $f_Y = F_Y'$. Here are the details:

$$F_Y(a) = \mathrm{P}\{Y \leq a\} = \mathrm{P}\left\{-\ln X \leq a\right\}.$$

Now, $-\ln(x)$ is a decreasing function. Therefore, $-\ln(x) \leq a$ if and only if $x \geq e^{-a}$, and hence,

$$F_Y(a) = \mathrm{P}\left\{X \geq e^{-a}\right\} = 1 - F_X(e^{-a}).$$

Consequently,

$$f_Y(a) = -f_X(e^{-a})\frac{d}{da}(e^{-a}) = e^{-a}f_X(e^{-a}).$$

Now recall that $f_X(u) = 1$ if $0 \leq u \leq 1$ and $f_X(u) = 0$ otherwise. Now e^{-a} is between zero and one if and only if $a \geq 0$. Therefore,

$$f_X(e^{-a}) = \begin{cases} 1 & \text{if } a \geq 0, \\ 0 & \text{if } a < 0. \end{cases}$$

It follows then that

$$f_Y(a) = \begin{cases} e^{-a} & \text{if } a \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $-\ln X$ has an exponential density with parameter $\lambda = 1$. More generally, if $\lambda > 0$ is fixed, then $-(1/\lambda)\ln X$ has an exponential density with parameter λ.

Example 22.4. Suppose X has density f_X. Then let us find the density function of $Y = X^2$. Again, we seek to first compute F_Y. Now, for all $a > 0$,

$$F_Y(a) = \mathrm{P}\{X^2 \le a\} = \mathrm{P}\left\{-\sqrt{a} \le X \le \sqrt{a}\right\} = F_X\left(\sqrt{a}\right) - F_X\left(-\sqrt{a}\right).$$

Differentiate $[d/da]$ to find that

$$f_Y(a) = \frac{f_X\left(\sqrt{a}\right) + f_X\left(-\sqrt{a}\right)}{2\sqrt{a}}$$

On the other hand, $f_Y(a) = 0$ if $a \le 0$. For example, consider the case that X is standard normal. Then,

$$f_{X^2}(a) = \begin{cases} \dfrac{e^{-a}}{\sqrt{2\pi a}} & \text{if } a > 0, \\ 0 & \text{if } a \le 0. \end{cases}$$

Or if X is Cauchy, then

$$f_{X^2}(a) = \begin{cases} \dfrac{1}{\pi\sqrt{a}(1+a)} & \text{if } a > 0, \\ 0 & \text{if } a \le 0. \end{cases}$$

Example 22.5. Suppose $\mu \in \mathbf{R}$ and $\sigma > 0$ are fixed constants, and define $Y = \mu + \sigma X$. Find the density of Y in terms of that of X. Once again,

$$F_Y(a) = \mathrm{P}\left\{\mu + \sigma X \le a\right\} = \mathrm{P}\left\{X \le \frac{a-\mu}{\sigma}\right\} = F_X\left(\frac{a-\mu}{\sigma}\right).$$

Therefore,

$$f_Y(a) = \frac{1}{\sigma} f_X\left(\frac{a-\mu}{\sigma}\right).$$

For example, if X is standard normal, then

$$f_{\mu+\sigma X}(a) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

This is the socalled $N(\mu\,,\sigma^2)$ density.

Example 22.6. Suppose X is uniformly distributed on $(0\,,1)$, and define

$$Y = \begin{cases} 0 & \text{if } 0 \le X < \frac{1}{3}, \\ 1 & \text{if } \frac{1}{3} \le X < \frac{2}{3}, \\ 2 & \text{if } \frac{2}{3} \le X < 1. \end{cases}$$

Then, Y is a discrete random variable with mass function,

$$f_Y(x) = \begin{cases} \frac{1}{3} & \text{if } x = 0,\ 1, \text{ or } 2, \\ 0 & \text{otherwise.} \end{cases}$$

For instance, in order to compute $f_Y(1)$ we note that

$$f_Y(1) = \mathrm{P}\left\{\frac{1}{3} \le X < \frac{2}{3}\right\} = \int_{1/3}^{2/3} \underbrace{f_X(y)}_{\equiv 1}\, dy = \frac{1}{3}.$$