Lecture 21

1. Continuous Random Variables

Definition 21.1. We say that X is a *continuous* random variable with *density function* f if f is a piecewise continuous nonnegative function, and for all real numbers x,

$$\mathrm{P}\{X \le x\} = \int_{-\infty}^{x} f(y)\,dy.$$

In this case,

$$F(x) = \mathrm{P}\{X \le x\} = \int_{-\infty}^{x} f(y)\,dy$$

defines the *distribution function* of X.

Some basic properties:

(1) We have $F(\infty) - F(-\infty) = \int_{-\infty}^{\infty} f(y)\,dy = 1$.

(2) Because f is integrable and nonnegative, for all real numbers x we have

$$F(x+h) - F(x) = \int_{x}^{x+h} f(y)\,dy \to 0 \qquad \text{as } h \searrow 0.$$

But the left-most term is $\mathrm{P}\{x < X \le x+h\}$. Therefore, by Rule 4 of probabilities,

$$\mathrm{P}\{X = x\} = F(x) - F(x-) = 0 \qquad \text{for all } x.$$

(3) If f is continuous at x, then by the fundamental theorem of calculus,

$$F'(x) = f(x).$$

This shows that $F' = f$ at all but at most countably-many points.

For examples, we merely need to construct *any* f such that $f(x) \geq 0$ and $\int_{-\infty}^{x} f(y)\,dy = 1$, together with the property that f is continuous piecewise. Here are some standard examples.

Example 21.2 (Uniform density)**.** If $a < b$ are fixed, then the uniform density on $(a\,,b)$ is the function

$$f(x) = \begin{cases} \dfrac{1}{b-a} & \text{if } a \leq x \leq b, \\ 0 & \text{otherwise.} \end{cases}$$

In this case, we can compute the distribution function as follows:

$$F(x) = \begin{cases} 0 & \text{if } x < a, \\ \dfrac{x}{b-a} & \text{if } a \leq x \leq b, \\ 1 & \text{if } x > b. \end{cases}$$

Example 21.3 (Exponential densities)**.** Let $\lambda > 0$ be fixed. Then

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0, \\ 0 & \text{if } x < 0 \end{cases}$$

is a density, and is called the *exponential density with parameter* λ. It is not hard to see that

$$F(x) = \begin{cases} 1 - e^{-\lambda x} & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Example 21.4 (The Cauchy density)**.** Define for all real numbers x,

$$f(x) = \frac{1}{\pi}\,\frac{1}{1+x^2}.$$

Because

$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2},$$

we have

$$\int_{-\infty}^{\infty} f(y)\,dy = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{1+y^2}\,dy = \frac{1}{\pi}\left[\arctan(\infty) - \arctan(-\infty)\right] = 1.$$

Also,

$$\begin{aligned} F(x) &= \frac{1}{\pi}\int_{-\infty}^{x} f(y)\,dy = \frac{1}{\pi}\left[\arctan(x) - \arctan(-\infty)\right] \\ &= \frac{1}{\pi}\arctan(x) + \frac{1}{2} \qquad \text{for all real } x. \end{aligned}$$