Lecture 18

1. The distribution of the sum of two independent random variables, continued

Recall that if X and Y are independent, then

$$f_{X+Y}(z) = \sum_x f_X(x) f_Y(z-x).$$

Now we work out three examples of this. [We have seen another already at the end of Lecture 17.]

Example 18.1. Suppose $X = \pm 1$ with probability 1/2 each; and $Y = \pm 2$ with probability 1/2 each. Then,

$$f_{X+Y}(z) = \begin{cases} 1/4 & \text{if } z = 3, -3, 1, -1, \\ 0 & \text{otherwise.} \end{cases}$$

Example 18.2. Let X and Y denote two independent geometric(p) random variables with the same parameter $p \in (0\,,1)$. What is the mass function of $X+Y$? If $z = 2, 3, \ldots,$ then

$$\begin{aligned} f_{X+Y}(z) &= \sum_x f_X(x) f_Y(z-x) = \sum_{x=1}^{\infty} pq^{x-1} f_Y(z-x) \\ &= \sum_{x=1}^{z+1} pq^{x-1} pq^{z-x-1} = p^2 \sum_{x=1}^{z+1} q^{z-2} = (z+1)p^2 q^{z-2}. \end{aligned}$$

Else, $f_{X+Y}(z) = 0$. This shows that $X+Y$ is a negative binomial. Can you deduce this directly, and by other means?

Example 18.3. If $X = \text{bin}(n\,, p)$ and $Y = \text{bin}(m\,, p)$ for the same parameter $p \in (0\,,1)$, then what is the distribution of $X+Y$? If $z = 0, 1, \ldots, n+m$, then

$$\begin{aligned} f_{X+Y}(z) &= \sum_x f_X(x) f_Y(z-x) = \sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x} \\ &= \sum_{\substack{0 \le x \le n \\ 0 \le z-x \le m}} \binom{n}{x} p^x q^{n-x} \binom{m}{z-x} p^{z-x} q^{m-(z-x)} \\ &= p^z q^{m+n-z} \sum_{\substack{0 \le x \le n \\ z-m \le x \le z}} \binom{n}{x}\binom{m}{z-x}. \end{aligned}$$

[The sum is over all integers x such that x is between 0 and n, *and* x is also betweem $z-m$ and m.] For other values of z, $f_{X+Y}(z) = 0$.

Equivalently, we can write for all $z = 0, \ldots, n+m$,

$$f_{X+Y}(z) = \binom{n+m}{z} p^z q^{m+n-z} \sum_{\substack{0 \le x \le n \\ z-m \le x \le z}} \frac{\binom{n}{x}\binom{m}{z-x}}{\binom{n+m}{z}}.$$

Thus, if we showed that the sum is one, then $X+Y = \text{bin}(n+m\,, p)$. In order to show that the sum is one consider an urn that has n white balls and m black balls. We choose z balls at random, without replacement. The probability that we obtain exactly x white and $z-x$ black is precisely,

$$\frac{\binom{n}{x}\binom{m}{z-x}}{\binom{n+m}{z}}.$$

Therefore, if we add this probability over all possible values of x we should get one. This does the job.

Can you find a direct way to prove that $X+Y = \text{bin}(n+m\,, p)$?

2. Transformations of a mass function

Let f denote the mass function of a random variable. For technical reasons, one often "transforms" f into a new function which is easier to analyze some times. The transformation can be fairly arbitrary, but it should be possible, in principle, to compute f from that transformation as well. In this way, the computations for the transform will often yield useful computations for the original mass function. [We do this only when it is very hard to work with the mass function directly.]

In this course we will study only two transformations: The generating function, and the moment generating function.

2.1. The generating function. If X is integer valued, then its *generating function* [also known as the "probability generating function," or p.g.f., for short] G is the function

$$G(s) = \sum_k s^k f(s) \qquad \text{for all } s \in (-1\,,1).$$

That is, we start with some mass function f, and transform it into another function—the generating function—G. Note that

$$G(s) = \mathrm{E}[s^X].$$

This is indeed a useful transformation. Indeed,

Theorem 18.4 (Uniqueness)**.** *If $G_X(s) = G_Y(s)$ for all $s \in (-1\,,1)$, then $f_X = f_Y$.*

In order to go from G to f we need a lot of examples. In this course, we will work out a few. Many more are known.

Example 18.5. Suppose X is uniformly distributed on $\{-n\,,\ldots,m\}$, where n and m are positive integers. This means that $f(x) = 1/(m+n+1)$ if $x = -n,\ldots,m$ and $f(x) = 0$ otherwise. Consequently, for all $s \in (-1\,,1)$,

$$\begin{aligned} G(s) &= \sum_{x=-n}^{m} \frac{s^x}{n+m+1} = \frac{1}{n+m+1}\sum_{x=-n}^{m} s^x \\ &= \frac{s^{-n} - s^{m+1}}{(n+m+1)(1-s)}, \end{aligned}$$

using facts about geometric series.

Example 18.6. Suppose

$$G(s) = \frac{(\alpha-1)s}{\alpha - s} \qquad \text{for all } s \in (-1\,,1),$$

where $\alpha > 1 > 0$. I claim that G is a p.g.f. The standard way to do this is to expand G into a Taylor expansion. Define

$$h(s) = \frac{1}{\alpha - s} = (\alpha - s)^{-1}.$$

Then, $h'(s) = (\alpha - s)^{-2}$, $h''(s) = 2(\alpha - s)^{-3}$, etc., and in general,

$$h^{(n)}(s) = n!(\alpha - s)^{-(n+1)}.$$

According to the Taylor-MacLaurin expansion of h,

$$h(s) = \sum_{n=0}^{\infty} \frac{1}{n!} s^n h^{(n)}(0).$$

Note that $h^{(n)}(0) = \alpha^{-1} n! \alpha^{-n}$. Therefore, as long as $0 < s/\alpha < 1$,

$$\frac{1}{\alpha - s} = \frac{1}{\alpha} \sum_{n=0}^{\infty} \left(\frac{s}{\alpha}\right)^n.$$

In particular,

$$G(s) = \frac{(\alpha-1)s}{\alpha} \sum_{n=0}^{\infty} s^n (1/\alpha)^n = \frac{\alpha - 1}{\alpha} \sum_{k=1}^{\infty} s^k (1/\alpha)^{k-1}.$$

By the uniqueness theorem,

$$f(k) = \begin{cases} \dfrac{\alpha-1}{\alpha} \left(\dfrac{1}{\alpha}\right)^{k-1} & \text{if } k = 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, in fact, $X = \text{geometric}(1/\alpha)$.