Lecture 17

1. Wrap-up of Lecture 16

Proof of Lemma 16.8. It suffices to prove that

$$\mathrm{E}\,(X_1 + \cdots + X_n) = n\mu$$
$$\mathrm{Var}\,(X_1 + \cdots + X_n) = n\sigma^2.$$

We prove this by induction. Indeed, this is obviously true when $n = 1$. Suppose it is OK for all integers $\leq n-1$. We prove it for n.

$$\begin{aligned}\mathrm{E}\,(X_1 + \cdots + X_n) &= \mathrm{E}\,(X_1 + \cdots + X_{n-1}) + \mathrm{E}X_n \\ &= (n-1)\mu + \mathrm{E}X_n,\end{aligned}$$

by the induction hypothesis. Because $\mathrm{E}X_n = \mu$, the preceding is equal to $n\mu$, as planned. Now we verify the more interesting variance computation.

Once again, we assume the assertion holds for all integers $\leq n-1$, and strive to check it for n.

Define

$$Y = X_1 + \cdots + X_{n-1}.$$

Because Y is independent of X_n, $\mathrm{Cov}(Y, X_n) = 0$. Therefore, by Lecture 15,

$$\begin{aligned}\mathrm{Var}\,(X_1 + \cdots + X_n) &= \mathrm{Var}(Y + X_n) \\ &= \mathrm{Var}(Y) + \mathrm{Var}(X_n) + \mathrm{Cov}(Y, X_n) \\ &= \mathrm{Var}(Y) + \mathrm{Var}(X_n).\end{aligned}$$

We know that $\mathrm{Var}(X_n) = \sigma^2$, and by the induction hypothesis, $\mathrm{Var}(Y) = (n-1)\sigma^2$. The result follows. □

2. Conditioning

2.1. Conditional mass functions. For all y, define the conditional mass function of X given that $Y = y$ as

$$\begin{aligned} f_{X|Y}(x\,|\,y) = \mathrm{P}\left(X = x \,\middle|\, Y = y\right) &= \frac{\mathrm{P}\{X = x\,, Y = y\}}{\mathrm{P}\{Y = y\}} \\ &= \frac{f(x\,,y)}{f_Y(y)}, \end{aligned} \tag{16}$$

provided that $f_Y(y) > 0$.

As a function in x, $f_{X|Y}(x\,|\,y)$ is a probability mass function. That is:

(1) $0 \le f_{X|Y}(x\,|\,y) \le 1$;

(2) $\sum_x f_{X|Y}(x\,|\,y) = 1$.

Example 17.1 (Example 14.2, Lecture 14, continued)**.** In this example, the joint mass function of (X, Y), and the resulting marginal mass functions, were given by the following:

$x \setminus y$	**0**	**1**	**2**	f_X
0	16/36	8/36	1/36	25/36
1	8/36	2/36	0	10/36
2	1/36	0	0	1/36
f_Y	25/36	10/36	1/36	**1**

Let us calculate the conditional mass function of X, given that $Y = 1$:

$$f_{X|Y}(0\,|\,1) = \frac{f(0\,,1)}{f_Y(1)} = \frac{8}{10}$$

$$f_{X|Y}(1\,|\,1) = \frac{f(1\,,1)}{f_Y(1)} = \frac{2}{10}$$

$$f_{X|Y}(x\,|\,1) = 0 \text{ for other values of } x.$$

Similarly,

$$f_{X|Y}(0\,|\,0) = \frac{16}{25}$$

$$f_{X|Y}(1\,|\,0) = \frac{8}{25}$$

$$f_{X|Y}(2\,|\,0) = \frac{1}{25}$$

$$f_{X|Y}(x\,|\,0) = 0 \text{ for other values of } x,$$

and

$$f_{X|Y}(0\,|\,2) = 1$$

$$f_{X|Y}(x\,|\,2) = 0 \text{ for other values of } x.$$

2.2. Conditional expectations. Define conditional expectations, as we did ordinary expectations. But use conditional probabilities in place of ordinary probabilities, viz.,

$$\mathrm{E}(X \mid Y=y)=\sum_x x f_{X|Y}(x \mid y). \tag{17}$$

Example 17.2 (Example 17.1, continued)**.** Here,

$$\mathrm{E}(X \mid Y=1)=\left(0\times\frac{8}{10}\right)+\left(1\times\frac{2}{10}\right)=\frac{2}{10}=\frac{1}{5}.$$

Similarly,

$$\mathrm{E}(X \mid Y=0)=\left(0\times\frac{16}{25}\right)+\left(1\times\frac{8}{25}\right)+\left(2\times\frac{1}{25}\right)=\frac{10}{25}=\frac{2}{5},$$

and

$$\mathrm{E}(X \mid Y=2)=0.$$

Note that $\mathrm{E}(X)=12/36=1/3$, which is none of the preceding. If you know that $Y=0$, then your best bet for X is $2/5$. But if you have no extra knowledge, then your best bet for X is $1/3$.

However, let us note the Bayes's formula in action:

$$\begin{aligned}
&\mathrm{E}(X)\\
&=\mathrm{E}(X \mid Y=0)\mathrm{P}\{Y=0\}+\mathrm{E}(X \mid Y=1)\mathrm{P}\{Y=1\}+\mathrm{E}(X \mid Y=2)\mathrm{P}\{Y=2\}\\
&=\left(\frac{2}{5}\times\frac{25}{36}\right)+\left(\frac{1}{5}\times\frac{10}{36}\right)+\left(0\times\frac{1}{36}\right)\\
&=\frac{12}{36},
\end{aligned}$$

as it should be.

3. Sums of independent random variables

Theorem 17.3. *If X and Y are independent, then*

$$f_{X+Y}(z)=\sum_x f_X(x)f_Y(z-x).$$

Proof. We note that $X+Y=z$ if $X=x$ for some x and $Y=z-x$ for that x. For example, suppose X is integer-valued and ≥ 1. Then $\{X+Y=z\}=\cup_{x=1}^{\infty}\mathrm{P}\{X=x\,,Y=z-x\}$. In general,

$$f_{X+Y}(z)=\sum_x \mathrm{P}\{X=x\,,Y=z-x\}=\sum_x \mathrm{P}\{X=x\}\mathrm{P}\{Y=z-x\}.$$

This is the desired result. □

Example 17.4. Suppose $X = \text{Poisson}(\lambda)$ and $Y = \text{Poisson}(\gamma)$ are independent. Then, I claim that $X + Y = \text{Poisson}(\lambda + \gamma)$. We verify this by directly computing as follows: The possible values of $X + Y$ are $0, 1, \ldots$. Let $z = 0, 1, \ldots$ be a possible value, and then check that

$$\begin{aligned}
f_{X+Y}(z) &= \sum_{x=0}^{\infty} f_X(x) f_Y(z-x) \\
&= \sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^x}{x!} f_Y(z-x) \\
&= \sum_{x=0}^{z} \frac{e^{-\lambda}\lambda^x}{x!} \frac{e^{-\gamma}\gamma^{z-x}}{(z-x)!} \\
&= \frac{e^{-(\lambda+\gamma)}}{z!} \sum_{x=0}^{z} \binom{z}{x} \lambda^x \gamma^{z-x} \\
&= \frac{e^{-(\lambda+\gamma)}}{z!} (\lambda+\gamma)^z,
\end{aligned}$$

thanks to the binomial theorem. For other values of z, it is easy to see that $f_{X+Y}(z) = 0$.