Lecture 14

Example 14.1 (St.-Petersbourg paradox, continued)**.** We continued with our discussion of the St.-Petersbourg paradox, and note that for all integers $N \geq 1$,

$$\begin{aligned}
g(N) &= g(1) + \sum_{k=2}^{N} \Big(g(k) - g(k-1) \Big) \\
&= g(1) + \sum_{k=1}^{N-1} \Big(g(k+1) - g(k) \Big) \\
&= g(1) + \sum_{k=1}^{N-1} \Big(-2 + g(k) - g(k-1) \Big) \\
&= g(1) - 2(N-1) + \sum_{k=1}^{N} \Big(g(k) - g(k-1) \Big) \\
&= g(1) - 2(N-1) + g(N).
\end{aligned}$$

If $g(1) < \infty$, then $g(1) = 2(N-1)$. But N is arbitrary. Therefore, $g(1)$ cannot be finite; i.e.,

$$\mathrm{E}(T_1) = \infty.$$

This shows also that $\mathrm{E}(T_x) = \infty$ for all $x \geq 1$, because for example $T_2 \geq 1 + T_1$! By symmetry, $\mathrm{E}(T_x) = \infty$ if x is a negative integer as well.

1. Joint distributions

If X and Y are two discrete random variables, then their *joint mass function* is

$$f(x\,,y) = \mathrm{P}\{X = x\,, Y = y\}.$$

We might write $f_{X,Y}$ in place of f in order to emphasize the dependence on the two random variables X and Y.

Here are some properties of $f_{X,Y}$:

- $f(x\,,y) \geq 0$ for all x, y;
- $\sum_x \sum_y f(x\,,y) = 1$;
- $\sum_{(x,y)\in C} f(x\,,y) = \mathrm{P}\{(X\,,Y) \in C\}$.

Example 14.2. You roll two fair dice. Let X be the number of 2s shown, and Y the number of 4s. Then X and Y are discrete random variables, and

$$\begin{aligned} f(x\,,y) &= \mathrm{P}\{X = x\,, Y = y\} \\ &= \begin{cases} \frac{1}{36} & \text{if } x = 2 \text{ and } y = 0, \\ \frac{1}{36} & \text{if } x = 0 \text{ and } y = 2, \\ \frac{2}{36} & \text{if } x = y = 1, \\ \frac{8}{36} & \text{if } x = 0 \text{ and } y = 1, \\ \frac{8}{36} & \text{if } x = 1 \text{ and } y = 0, \\ \frac{16}{36} & \text{if } x = y = 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Some times it helps to draw up a table of "joint probabilities":

$\boldsymbol{x \setminus y}$	**0**	**1**	**2**
0	16/36	8/36	1/36
1	8/36	2/36	0
2	1/36	0	0

From this we can also calculate f_X and f_Y. For instance,

$$f_X(1) = \mathrm{P}\{X = 1\} = f(1\,,0) + f(1\,,1) = \frac{10}{36}.$$

In general, you compute the row sums (f_X) and put them in the margin; you do the same with the column sums (f_Y) and put them in the bottom row. In this way, you obtain:

$\boldsymbol{x \setminus y}$	**0**	**1**	**2**	$\boldsymbol{f_X}$
0	16/36	8/36	1/36	25/36
1	8/36	2/36	0	10/36
2	1/36	0	0	1/36
$\boldsymbol{f_Y}$	25/36	10/36	1/36	**1**

The "**1**" designates the right-most column sum (which should be one), and/or the bottom-row sum (which should also be one). This is also the sum of the elements of the table (which should also be one).

En route we have discovered the next result, as well.

Theorem 14.3. *For all x, y:*

(1) $f_X(x) = \sum_b f(x\,, b)$.
(2) $f_Y(y) = \sum_a f(a\,, y)$.

2. Independence

Definition 14.4. Let X and Y be discrete with joint mass function f. We say that X and Y are *independent* if for all x, y,

$$f(x\,, y) = f_X(x) f_Y(y).$$

- Suppose A and B are two sets, and X and Y are independent. Then,
$$\begin{aligned} \mathrm{P}\{X \in A\,, Y \in B\} &= \sum_{x \in A} \sum_{y \in B} f(x\,, y) \\ &= \sum_{x \in A} f_X(x) \sum_{y \in B} f_Y(y) \\ &= \mathrm{P}\{X \in A\} \mathrm{P}\{Y \in B\}. \end{aligned}$$
- Similarly, if h and g are functions, then $h(X)$ and $g(Y)$ are independent as well.
- All of this makes sense for more than 2 random variables as well.

Example 14.5 (Example 14.2, continued)**.** Note that in this example, X and Y are not independent. For instance,

$$f(1\,, 2) = 0 \neq f_X(1) f_Y(2) = \frac{10}{36} \times \frac{1}{36}.$$

Now, let us find the distribution of $Z = X + Y$. The possible values are 0, 1, and 2. The probabilities are

$$\begin{aligned} &f_Z(0) = f_{X,Y}(0\,, 0) = \frac{16}{36} \\ &f_Z(1) = f_{X,Y}(1\,, 0) + f_{X,Y}(0\,, 1) = \frac{8}{36} + \frac{8}{36} = \frac{16}{36} \\ &f_Z(2) = f_{X,Y}(0\,, 2) + f_{X,Y}(2\,, 0) + f_{X,Y}(1\,, 1) = \frac{1}{36} + \frac{1}{36} + \frac{2}{36} = \frac{4}{36}. \end{aligned}$$

That is,

$$f_Z(x) = \begin{cases} \frac{16}{36} & \text{if } x = 0 \text{ or } 1, \\ \frac{4}{36} & \text{if } x = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Example 14.6. *Let* $X = \text{geometric}(p_1)$ *and* $Y = \text{geometric}(p_2)$ *be independent. What is the mass function of* $Z = \min(X\,, Y)$*?*

Recall from Lecture 9 that $\mathrm{P}\{X \geq n\} = q_1^{n-1}$ and $\mathrm{P}\{Y \geq n\} = q_2^{n-1}$ for all integers $n \geq 1$. Therefore,

$$\begin{aligned} \mathrm{P}\{Z \geq n\} &= \mathrm{P}\{X \geq n\,, Y \geq n\} = \mathrm{P}\{X \geq n\}\mathrm{P}\{Y \geq n\} \\ &= (q_1 q_2)^{n-1}, \end{aligned}$$

as long as $n \geq 1$ is an integer. Because $\mathrm{P}\{Z \geq n\} = \mathrm{P}\{Z = n\} + \mathrm{P}\{Z \geq n+1\}$, for all integers $n \geq 1$,

$$\begin{aligned} \mathrm{P}\{Z = n\} &= \mathrm{P}\{Z \geq n\} - \mathrm{P}\{Z \geq n+1\} = (q_1 q_2)^{n-1} - (q_1 q_2)^n \\ &= (q_1 q_2)^{n-1} (1 - q_1 q_2)\,. \end{aligned}$$

Else, $\mathrm{P}\{Z = n\} = 0$. Thus, $Z = \text{geometric}(p)$, where $p = 1 - q_1 q_2$.