Lecture 12

By the Cauchy–Schwarz inequality, if $\mathrm{E}(X^2) < \infty$, then $\mathrm{E}X$ is well defined and finite as well. In that case, the *variance* of X is defined as

$$\mathrm{var}(X) = \mathrm{E}(X^2) - |\mathrm{E}X|^2 .$$

In order to understand why this means anything, note that

$$\begin{aligned}
\mathrm{E}\left[(X - \mathrm{E}X)^2\right] &= \mathrm{E}\left[X^2 - 2X\mathrm{E}X + (\mathrm{E}X)^2\right] = \mathrm{E}(X^2) - 2\mathrm{E}(X)\mathrm{E}(X) + (\mathrm{E}X)^2 \\
&= \mathrm{E}(X^2) - |\mathrm{E}X|^2 \\
&= \mathrm{var}(X).
\end{aligned}$$

Thus:

(1) We predict the as-yet-unseen value of X by the nonrandom number $\mathrm{E}X$;

(2) $\mathrm{var}(X)$ is the expected squared-error in this prediction. Note that $\mathrm{var}(X)$ is also a nonrandom number.

1. Example 1

If $X = \mathrm{Bin}(n\,, p)$, then we have seen that $\mathrm{E}X = np$ and $\mathrm{E}(X^2) = (np)^2 + npq$. Therefore, $\mathrm{var}(X) = npq$.

2. Example 2

Suppose X has mass function

$$f(x) = \begin{cases} 1/4 & \text{if } x = 0, \\ 3/4 & \text{if } x = 1, \\ 0 & \text{otherwise.} \end{cases}$$

We saw in Lecture 11 that $\mathrm{E}X = 3/4$. Now we compute the variance by first calculating

$$\mathrm{E}(X^2) = \left(0^2 \times \frac{1}{4}\right) + \left(1^2 \times \frac{3}{4}\right) = \frac{3}{4}.$$

Thus,

$$\mathrm{var}(X) = \frac{3}{4} - \left(\frac{3}{4}\right)^2 = \frac{3}{4}\left(1 - \frac{3}{4}\right) = \frac{3}{16}.$$

3. Example 3

Let n be a fixed positive integer, and X takes any of the values $1, \dots, n$ with equal probability. Then, $f(x) = 1/n$ if $x = 1, \dots, n$; $f(x) = 0$, otherwise. Let us calculate the first two "moments" of X.[1] In this way, we obtain the mean and the variance of X.

The first moment is the expectation, or the mean, and is

$$\mathrm{E}X = \sum_{k=1}^{n} \frac{k}{n} = \frac{1}{n} \times \frac{(n+1)n}{2} = \frac{n+1}{2}.$$

In order to compute $\mathrm{E}(X^2)$ we need to know the algebraic identity:

$$\sum_{k=1}^{n} k^2 = \frac{(2n+1)(n+1)n}{6}. \tag{10}$$

This is proved by induction: For $n = 1$ it is elementary. Suppose it is true for $n - 1$. Then write

$$\sum_{k=1}^{n} k^2 = \sum_{k=1}^{n-1} k^2 + n^2 = \frac{(2(n-1)+1)(n-1+1)(n-1)}{6} + n^2,$$

thanks to the induction hypothesis. Simplify to obtain

$$\begin{aligned}\sum_{k=1}^{n} k^2 &= \frac{(2n-1)n(n-1)}{6} + n^2 = \frac{(2n-1)(n^2-n)}{6} + n^2 \\ &= \frac{2n^3 - 3n^2 + n}{6} + \frac{6n^2}{6} = \frac{2n^3 + 3n^2 + n}{6} = \frac{n(2n^2 + 3n + 1)}{6},\end{aligned}$$

which easily yields (10).

Thus,

$$\mathrm{E}(X^2) = \sum_{k=1}^{n} \frac{k^2}{n} = \frac{1}{n} \times \frac{(2n+1)(n+1)n}{6} = \frac{(2n+1)(n+1)}{6}.$$

[1] It may help to recall that the pth moment of X is $\mathrm{E}(X^p)$.

Therefore,

$$\begin{aligned}\operatorname{var}(X) &= \frac{(2n+1)(n+1)}{6} - \left(\frac{n+1}{2}\right)^2 = \frac{2n^2+3n+1}{6} - \frac{n^2+2n+1}{4}\\ &= \frac{4n^2+6n+2}{12} - \frac{3n^2+6n+3}{12}\\ &= \frac{n^2-1}{12}.\end{aligned}$$

4. Example 4

Suppose $X = \text{Poisson}(\lambda)$. We saw in Lecture 10 that $\mathrm{E}X = \lambda$. In order to compute $\mathrm{E}(X^2)$, we first compute $\mathrm{E}[X(X-1)]$ and find that

$$\begin{aligned}\mathrm{E}[X(X-1)] &= \sum_{k=0}^{\infty} k(k-1)\frac{e^{-\lambda}\lambda^k}{k!} = \sum_{k=2}^{\infty} \frac{e^{-\lambda}\lambda^k}{(k-2)!}\\ &= \lambda^2 \sum_{k=2}^{\infty} \frac{e^{-\lambda}\lambda^{k-2}}{(k-2)!}.\end{aligned}$$

The sum is equal to one; change variables ($j = k-2$) and recognize the jth term as the probability that $\text{Poisson}(\lambda) = j$. Therefore,

$$\mathrm{E}[X(X-1)] = \lambda^2.$$

Because $X(X-1) = X^2 - X$, the left-hand side is $\mathrm{E}(X^2) - \mathrm{E}X = \mathrm{E}(X^2) - \lambda$. Therefore,

$$\mathrm{E}(X^2) = \lambda^2 + \lambda.$$

It follows that

$$\operatorname{var}(X) = \lambda.$$

5. Example 5

Suppose $f(x) = pq^{x-1}$ if $x = 1, 2, \ldots$; and $f(x) = 0$ otherwise. This is the Geometric(p) distribution. [The mass function for the first time to heads for a p-coin; see Lecture 8.] We have seen already that $\mathrm{E}X = 1/p$ (Lecture 10). Let us find a new computation for this fact, and then go on and find also the variance.

$$\begin{aligned}\mathrm{E}X &= \sum_{k=1}^{\infty} kpq^{k-1} = p\sum_{k=1}^{\infty} kq^{k-1}\\ &= p\,\frac{d}{dq}\left(\sum_{k=0}^{\infty} q^k\right) = p\,\frac{d}{dq}\left(\frac{1}{1-q}\right) = \frac{p}{(1-q)^2} = \frac{1}{p}.\end{aligned}$$

Next we compute $\mathrm{E}(X^2)$ by first finding

$$\begin{aligned}
\mathrm{E}[X(X-1)] &= \sum_{k=1}^{\infty} k(k-1)pq^{k-1} = \frac{p}{q}\sum_{k=1}^{\infty} k(k-1)q^{k-2} \\
&= pq\ \frac{d^2}{dq^2}\left(\sum_{k=0}^{\infty} q^k\right) = \frac{p}{q}\ \frac{d^2}{dq^2}\left(\frac{1}{1-q}\right) \\
&= pq\ \frac{d}{dq}\left(\frac{1}{(1-q)^2}\right) = pq\ \frac{2}{(1-q)^3} = \frac{2q}{p^2}.
\end{aligned}$$

Because $\mathrm{E}[X(X-1)] = \mathrm{E}(X^2) - \mathrm{E}X = \mathrm{E}(X^2) - (1/p)$, this proves that

$$\mathrm{E}(X^2) = \frac{2q}{p^2} + \frac{1}{p} = \frac{2q+p}{p^2} = \frac{2-p}{p^2}.$$

Consequently,

$$\mathrm{var}(X) = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2} = \frac{q}{p^2}.$$

For a wholly different solution, see Example (13) on page 124 of your text.