Lecture 10

1. (Cumulative) distribution functions

Let X be a discrete random variable with mass function f. The (cumulative) *distribution function* F of X is defined by

$$F(x) = \mathrm{P}\{X \le x\}.$$

Here are some of the properties of distribution functions:

(1) $F(x) \le F(y)$ whenever $x \le y$; therefore, F is non-decreasing.

(2) $1 - F(x) = \mathrm{P}\{X > x\}$.

(3) $F(b) - F(a) = \mathrm{P}\{a < X \le b\}$ for $a < b$.

(4) $F(x) = \sum_{y:\, y \le x} f(y)$.

(5) $F(\infty) = 1$ and $F(-\infty) = 0$. [Some care is needed]

(6) F is right-continuous. That is, $F(x+) = F(x)$ for all x.

(7) $f(x) = F(x) - F(x-)$ is the size of the jump [if any] at x.

Example 10.1. Suppose X has the mass function

$$f_X(x) = \begin{cases} \frac{1}{2} & \text{if } x = 0, \\ \frac{1}{2} & \text{if } x = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, X has equal chances of being zero and one. Define a new random variable $Y = 2X - 1$. Then, the mass function of Y is

$$f_Y(x) = f_X\left(\frac{x+1}{2}\right) = \begin{cases} \frac{1}{2} & \text{if } x = -1, \\ \frac{1}{2} & \text{if } x = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The procedure of this example actually produces a theorem.

Theorem 10.2. *If $Y = g(X)$ for a function g, then*

$$f_Y(x) = \sum_{z\colon g(z)=x} f_X(z).$$

2. Expectation

The *expectation* $\mathrm{E}X$ of a random variable X is defined formally as

$$\mathrm{E}X = \sum_x x f(x).$$

If X has infinitely-many possible values, then the preceding sum must be defined. This happens, for example, if $\sum_x |x| f(x) < \infty$. Also, $\mathrm{E}X$ is always defined [but could be $\pm\infty$] if $\mathrm{P}\{X \geq 0\} = 1$, or if $\mathrm{P}\{X \leq 0\} = 1$. The *mean* of X is another term for $\mathrm{E}X$.

Example 10.3. If X takes the values ± 1 with respective probabilities $1/2$ each, then $\mathrm{E}X = 0$.

Example 10.4. If $X = \mathrm{Bin}(n\,, p)$, then I claim that $\mathrm{E}X = np$. Here is why:

$$\begin{aligned}
\mathrm{E}X &= \sum_{k=0}^{n} k \overbrace{\binom{n}{k} p^k q^{n-k}}^{f(k)} \\
&= \sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!} p^k q^{n-k} \\
&= np \sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1} q^{(n-1)-(k-1)} \\
&= np \sum_{j=0}^{n-1} \binom{n-1}{j} p^j q^{(n-1)-j} \\
&= np,
\end{aligned}$$

thanks to the binomial theorem.

Example 10.5. Suppose $X = \text{Poiss}(\lambda)$. Then, I claim that $\mathrm{E}X = \lambda$. Indeed,

$$\begin{aligned}\mathrm{E}X &= \sum_{k=0}^{\infty} k\frac{e^{-\lambda}\lambda^k}{k!}\\ &= \lambda\sum_{k=1}^{\infty}\frac{e^{-\lambda}\lambda^{k-1}}{(k-1)!}\\ &= \lambda\sum_{j=0}^{\infty}\frac{e^{-\lambda}\lambda^j}{j!}\\ &= \lambda,\end{aligned}$$

because $e^{\lambda} = \sum_{j=0}^{\infty}\lambda^j/j!$, thanks to Taylor's expansion.

Example 10.6. Suppose X is negative binomial with parameters r and p. Then, $\mathrm{E}X = r/p$ because

$$\begin{aligned}\mathrm{E}X &= \sum_{k=r}^{\infty} k\binom{k-1}{r-1}p^r q^{k-r}\\ &= \sum_{k=r}^{\infty}\frac{k!}{(r-1)!(k-r)!}p^r q^{k-r}\\ &= r\sum_{k=r}^{\infty}\binom{k}{r}p^r q^{k-r}\\ &= \frac{r}{p}\sum_{k=r}^{\infty}\binom{k}{r}p^{r+1}q^{(k+1)-(r+1)}\\ &= \frac{r}{p}\sum_{j=r+1}^{\infty}\underbrace{\binom{j-1}{(r+1)-1}p^{r+1}q^{j-(r+1)}}_{\mathrm{P}\{\text{Negative binomial }(r+1\,,p)=j\}}\\ &= \frac{r}{p}.\end{aligned}$$

Thus, for example, $\mathrm{E}[\text{Geom}(p)] = 1/p$.

Finally, two examples to test the boundary of the theory so far.

Example 10.7 (A random variable with infinite mean)**.** Let X be a random variable with mass function,

$$f(x) = \begin{cases}\dfrac{1}{Cx^2} & \text{if } x = 1, 2, \ldots,\\ 0 & \text{otherwise,}\end{cases}$$

where $C = \sum_{j=1}^{\infty}(1/j^2)$. Then,

$$\mathrm{E}X = \sum_{j=1}^{\infty} j \cdot \frac{1}{Cj^2} = \infty.$$

But $\mathrm{P}\{X < \infty\} = \sum_{j=1}^{\infty} 1/(Cj^2) = 1$.

Example 10.8 (A random variable with an undefined mean)**.** Let X be a random with mass function,

$$f(x) = \begin{cases} \dfrac{1}{Dx^2} & \text{if } x = \pm 1, \pm 2, \ldots, \\ 0 & \text{otherwise,} \end{cases}$$

where $D = \sum_{j \in \mathbf{Z} \setminus \{0\}}(1/j^2)$. Then, $\mathrm{E}X$ is undefined. If it were defined, then it would be

$$\lim_{n,m\to\infty} \left(\sum_{j=-m}^{-1} \frac{j}{Dj^2} + \sum_{j=1}^{n} \frac{j}{Dj^2} \right) = \frac{1}{D} \lim_{n,m\to\infty} \left(\sum_{j=-m}^{-1} \frac{1}{j} + \sum_{j=1}^{n} \frac{1}{j} \right).$$

But the limit does not exist. The rough reason is that if N is large, then $\sum_{j=1}^{N}(1/j)$ is very nearly $\ln N$ plus a constant (Euler's constant). "Therefore," if n, m are large, then

$$\left(\sum_{j=-m}^{-1} \frac{1}{j} + \sum_{j=1}^{n} \frac{1}{j} \right) \approx -\ln m + \ln n = \ln\left(\frac{n}{m}\right).$$

If $n = m \to \infty$, then this is zero; if $m \gg n \to \infty$, then this goes to $-\infty$; if $n \gg m \to \infty$, then it goes to $+\infty$.