Lecture 1

1. The sample space, events, and outcomes

- Need a math model for describing "random" events that result from performing an "experiment."
- Ω denotes a sample space. We think of the elements of Ω as "outcomes" of the experiment.
- $\mathscr{F}$ is a collection of subsets of Ω; elements of $\mathscr{F}$ are called "events." We wish to assign a "probability" $\mathrm{P}(A)$ to every $A \in \mathscr{F}$. When Ω is finite, $\mathscr{F}$ is always taken to be the collection of all subsets of Ω.

Example 1.1. Roll a six-sided die; what is the probability of rolling a six? First, write a sample space. Here is a natural one:

$$\Omega = \{1, 2, 3, 4, 5, 6\}.$$

In this case, Ω is finite and we want $\mathscr{F}$ to be the collection of all subsets of Ω. That is,

$$\mathscr{F} = \Big\{ \varnothing \,, \Omega \,, \{1\} \,, \ldots, \{6\} \,, \{1,2\} \,, \ldots, \{1,6\} \,, \ldots, \{1,2,\ldots,6\} \Big\}.$$

Example 1.2. Toss two coins; what is the probability that we get two heads? A natural sample space is

$$\Omega = \Big\{ (H_1\,, H_2) \,, (H_1\,, T_2) \,, (T_1\,, H_2) \,, (T_1\,, T_2) \Big\}.$$

Once we have readied a sample space Ω and an event-space $\mathscr{F}$, we need to assign a probability to every event. This assignment cannot be made at whim; it has to satisfy some properties.

2. Rules of probability

Rule 1. $0 \leq \mathrm{P}(A) \leq 1$ for every event A.

Rule 2. $\mathrm{P}(\Omega) = 1$. "Something will happen with probability one."

Rule 3 (Addition rule)**.** If A and B are disjoint events [i.e., $A \cap B = \varnothing$], then the probability that at least one of the two occurs is the sum of the individual probabilities. More precisely put,

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B).$$

Lemma 1.3. *Choose and fix an integer $n \geq 1$. If $A_1, A_2, \ldots, A_n$ are disjoint events, then*

$$\mathrm{P}\left(\bigcup_{i=1}^{n} A_i\right) = \mathrm{P}(A_1) + \cdots + \mathrm{P}(A_n).$$

Proof. The proof uses *mathematical induction*.

Claim. If the assertion is true for $n-1$, then it is true for n.

The assertion is clearly true for $n = 1$, and it is true for $n = 2$ by Rule 3. Because it is true for $n = 2$, the Claim shows that the assertion holds for $n = 3$. Because it holds for $n = 3$, the Claim implies that it holds for $n = 4$, etc.

Proof of Claim. We can write $A_1 \cup \cdots \cup A_n$ as $A_1 \cup B$, where $B = A_2 \cup \cdots \cup A_n$. Evidently, A_1 and B are disjoint. Therefore, Rule 3 implies that $\mathrm{P}(A) = \mathrm{P}(A_1 \cup B) = \mathrm{P}(A_1) + \mathrm{P}(B)$. But B itself is a disjoint union of $n-1$ events. Therefore $\mathrm{P}(B) = \mathrm{P}(A_2) + \cdots + \mathrm{P}(A_n)$, thanks to the assumption of the Claim ["the induction hypothesis"]. This ends the proof. $\square$