Solutions to Midterm 4

Math 5010-1, Spring 2007, University of Utah

1. *Suppose X is a standard normal random variable. What is the density of $1/X^2$?*

   Solution: Let $Y = 1/X^2$ and note that for all $a > 0$,

$$\begin{aligned} F_Y(a) &= \mathrm{P}\left\{X^{-2} \le a\right\} = \mathrm{P}\left\{X^2 > a^{-1}\right\} \\ &= 1 - \mathrm{P}\left\{X^2 \le a^{-2}\right\} = 1 - \mathrm{P}\left\{|X| \le 1/\sqrt{a}\right\} \\ &= 1 - \left(F_X\left(1/\sqrt{a}\right) - F_X\left(-1/\sqrt{a}\right)\right) \\ &= 1 - \left(2F_X\left(1/\sqrt{a}\right) - 1\right) = 2 - 2F_X\left(1/\sqrt{a}\right). \end{aligned}$$

   Therefore,

$$\begin{aligned} f_Y(a) &= -2f_X\left(1/\sqrt{a}\right) \frac{d}{da}\left(\frac{1}{\sqrt{a}}\right) \\ &= -2f_X\left(1/\sqrt{a}\right) \times \left(-\frac{1}{2a^{3/2}}\right) \\ &= \frac{1}{a^{3/2}} f_X\left(1/\sqrt{a}\right) = \frac{1}{a^{3/2}\sqrt{2\pi}} \exp\left(-\frac{1}{2a}\right). \end{aligned}$$

   If $a \le 0$, then $f_Y(a) = 0$.

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2. *Compute the moment generating function of X, when X is a random variable with each of the following densities:*

   (a) $f(x) = \frac{1}{2}e^{-|x|}$.

   Solution: For all t,

$$\begin{aligned} M(t) &= \int_{-\infty}^{\infty} e^{tx} \frac{e^{-|x|}}{2}\, dx \\ &= \int_0^{\infty} \frac{e^{tx-x}}{2}\, dx + \int_{-\infty}^{0} \frac{e^{tx+x}}{2}\, dx \\ &= \int_0^{\infty} \frac{e^{tx-x}}{2}\, dx + \int_0^{\infty} \frac{e^{-tx-x}}{2}\, dx \\ &= \int_0^{\infty} \frac{e^{-x(1-t)}}{2}\, dx + \int_0^{\infty} \frac{e^{-x(1+t)}}{2}\, dx. \end{aligned}$$

If $t \geq 1$, then the first integral blows up. If $t \leq -1$, then the second integral blows up. Therefore, we have a finite $M(t)$ if and only if $|t| < 1$. In that case,

$$M(t) = \frac{1}{2(1-t)} + \frac{1}{2(1+t)} = \frac{1}{1-t^2}.$$

(b) $f(x) = 1 - |x|$ if $-1 < x < 1$; else, $f(x) = 0$.

Solution: For all real numbers t,

$$\begin{aligned} M(t) &= \int_{-1}^{1} (1 - |x|)\, e^{tx}\, dx = \int_{-1}^{1} e^{tx}\, dx - \int_{-1}^{1} |x| e^{tx}\, dx \\ &= \frac{e^t - e^{-t}}{t} - \int_0^1 x e^{tx}\, dx - \int_{-1}^{0} (-x) e^{tx}\, dx \\ &= \frac{e^t - e^{-t}}{t} - \int_0^1 x e^{tx}\, dx - \int_0^1 x e^{-tx}\, dx. \end{aligned}$$

Integrate by parts to find that for all real numbers s,

$$\begin{aligned} \int_0^1 x e^{sx}\, dx &= \int_0^1 \underbrace{u}_{x} \underbrace{v'}_{e^{sx}}\, dx = uv\Big|_0^1 - \int_0^1 \underbrace{v}_{\frac{1}{s}e^{sx}} \underbrace{u'}_{1}\, dx \\ &= \frac{e^s}{s} - \frac{1}{s}\int_0^1 e^{sx}\, dx = \frac{e^s}{s} - \frac{1 - e^s}{s^2}. \end{aligned}$$

Use this, once with $s = t$ and once with $s = -t$ to find an answer.

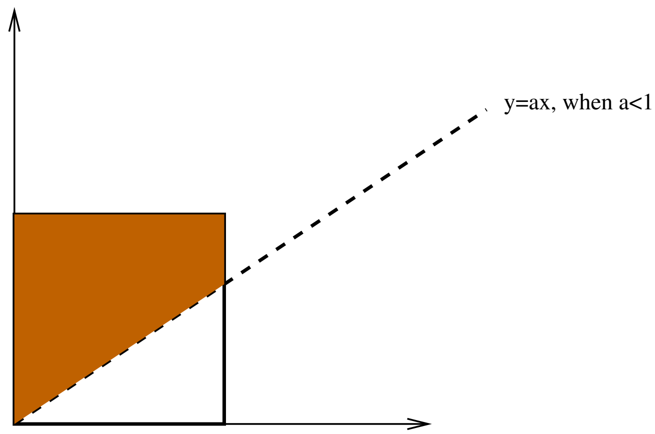

Figure 1: The case where $a < 1$

3. *Suppose* (X, Y) *is distributed uniformly on the square with corners at* $(0\,,0)$, $(1\,,0)$, $(0\,,1)$, *and* $(1\,,1)$. *Compute* $\mathrm{P}\{Y > aX\}$ *for all real numbers* a.

   Solution: By definition, the joint density function f of (X, Y) is

$$f(x\,,y) = \begin{cases} 1 & \text{if } 0 < x < 1 \text{ and } 0 < y < 1, \\ 0 & \text{otherwise.} \end{cases}$$

   Therefore, we see to find $\iint_{y>ax} f(x\,,t)\,dx\,dy$. The computation depends on the value of a.

   If $a = 1$, then $\mathrm{P}\{Y > X\} = \mathrm{P}\{Y > aX\} = 1/2$ by symmetry. Therefore, there are two cases to consider (Figures 1 and 2):

   If $a < 1$, then according to Figure 1,

$$\mathrm{P}\{Y > aX\} = \int_0^1 \int_{ax}^1 dy\,dx = \int_0^1 (1 - ax)\,dx = 1 - \frac{a}{2}.$$

   If, on the other hand, $a > 1$, then according to Figure 2,

$$\mathrm{P}\{Y > aX\} = \int_0^1 \int_0^{y/a} dx\,dy = \frac{1}{a}\int_0^1 y\,dy = \frac{1}{2a}.$$

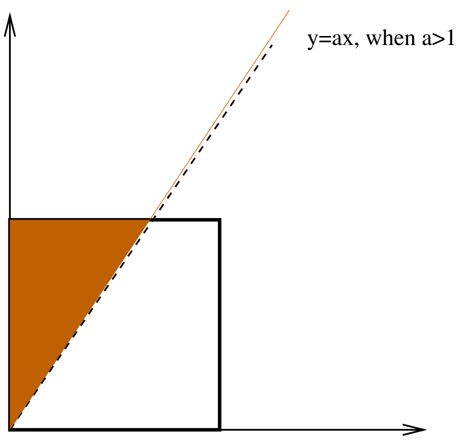

Figure 2: The case where $a > 1$

4. *Find* $\mathrm{E}X$*, when* (X, Y) *is jointly distributed with density*

$$f(x\,, y) = \begin{cases} 8xy & \text{if } 0 < y < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

You must do this by first computing f_X*.*

Solution: Integrate out the y variable first: If $0 < x < 1$, then

$$f_X(x) = \int_0^x 8xy\,dy = 4x^3.$$

Else, $f_X(x) = 0$. Consequently,

$$\mathrm{E}X = \int_0^1 4x^4\,dx = \frac{4}{5}.$$