

Name: Solutions

Midterm 3

Math 5010-1, Spring 2007, University of Utah

- ♣ This is a closed-book, closed-notes examination.
- ♣ This exam begins at 10:40 a.m. and ends at 11:30 a.m. sharp.
- ♡ This exam is made up of 4 questions for a total of 40 points.
- ◇ Write your answers clearly. If you show merely a numerical answer, then you will receive zero partial credit. Show and explain your work.
- ♣ Depending on the severity of the error(s), incorrect answers can lead to point deductions.
- ♣ Confine your work to this worksheet. You may use both sides of the paper.
- ♡ There is also an extra sheet of paper for you to write on if you wish.

1. (10 points total; 5 points each) Suppose X is a continuous random variable with density function $f(x) = Ce^{-|x|}$.

(a) (5 points) Find C .

$$1 = C \int_{-\infty}^{\infty} e^{-|x|} dx = 2C \int_0^{\infty} e^{-x} dx = 2C \Rightarrow \boxed{C = \frac{1}{2}}$$

(b) (5 points) Find the density function of $Y = |X|$.

$$F_Y(a) = P\{|X| \leq a\} = F_X(a) - F_X(-a)$$

$$\therefore \begin{cases} f_Y(a) = f_X(a) + f_X(-a) = \boxed{e^{-a}} & \dots \dots a > 0 \\ f_Y(a) = \boxed{0} & \dots \dots a \geq 0 \end{cases}$$

2. (10 points) Suppose $G(s) = a_1s + a_2s^2$ is the generating function of a random variable X , and $E(X) = 0$ and $\text{Var}(X) = 1$. Compute a_1 and a_2 .

$$\text{If } G(s) = a_1s + a_2s^2 \text{ then}$$

$$a_1 = P\{X=1\}, \text{ and}$$

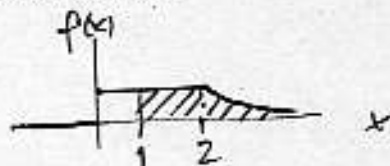
$$1 - a_1 = a_2 = P\{X=2\}.$$

$$E(X) = a_1 + 2a_2 \geq a_1 + a_2 \geq 1 \quad \textcircled{\times}$$

$\textcircled{\times} \Rightarrow E(X)$ cannot be zero.

3. (10 points) Compute $P\{X > 1\}$, where X is a continuous random variable with density

$$f(x) = \begin{cases} 1/4 & \text{if } 0 < x < 2, \\ x^{-2} & \text{if } x \geq 2. \end{cases}$$



$$\begin{aligned} P\{X > 1\} &= \int_1^2 \frac{1}{4} dx + \int_2^{\infty} \frac{1}{x^2} dx \\ &= \frac{1}{4} + \frac{1}{2} = \boxed{\frac{3}{4}} \end{aligned}$$

4. (10 points total) Suppose X is a random variable with mass function

$$f(k) = \begin{cases} C2^{-k} & \text{if } k = 1, 2, \dots \\ 0 & \text{otherwise.} \end{cases}$$

(a) (5 points) Compute C .

$$1 = \sum_{k=1}^{\infty} \frac{C}{2^k} = C \left[\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k - 1 \right] = C \left[\frac{1}{1 - 1/2} - 1 \right] = C$$

$$\boxed{C = 1}$$

(b) (5 points) Find the generating function of X .

$$G(t) = \sum_{k=1}^{\infty} t^k \frac{1}{2^k} = \sum_{k=1}^{\infty} \left(\frac{t}{2}\right)^k$$

$$= \begin{cases} \infty & \text{if } t \geq 2 \\ \frac{1}{1 - \frac{t}{2}} - 1 & \text{if } t < 2 \end{cases}$$

$$\frac{t/2}{1 - \frac{t}{2}} = \boxed{\frac{t}{2-t}}$$