Solutions to Midterm 2

1. *A pair of fair dice are cast, and the number of rolled dots, on each die, is recorded. Let X denote the sum of the two numbers. Find the probability mass function of X.*

   Solution. Here is the table for the mass function:

x	$f(x)$
2	1/36
3	2/36
4	3/36
5	4/36
6	5/36
7	6/36
8	5/36
9	4/36
10	3/36
11	2/36
12	1/36

   For example, let Y_j denote the result of the jth draw, and f_Y the joint mass function of (Y_1, Y_2), to find that

$$\begin{aligned} f(7) &= \mathrm{P}\{Y_1 + Y_2 = 7\} \\ &= f_Y(1,6) + f_Y(2,5) + f_Y(3,4) + f_Y(4,3) + f_Y(5,2) + f_Y(6,1) \\ &= \frac{1}{36} + \cdots + \frac{1}{36} = \frac{6}{36}. \end{aligned}$$

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2. *A fair coin is cast until the second tail appears. Let N denote the number of tosses needed to see the first head [correction: tail]. Find* $\mathrm{E}(N)$.

   Solution. Clearly, $N = \text{geometric}(1/2)$. That is, $\mathrm{P}\{N = n\} = (1/2)^n$ for $n = 1, 2, \ldots$. Hence,

$$\mathrm{E}(N) = \sum_{n=1}^{\infty} \frac{n}{2^n}.$$

   Recall that

$$\sum_{n=1}^{\infty} nr^n = r\sum_{n=1}^{\infty} nr^{n-1} = r\left(\sum_{n=0}^{\infty} r^n\right)' = r\left(\frac{1}{1-r}\right)' = \frac{r}{(1-r)^2}.$$

Apply this with $r = 1/2$ to find that

$$\mathrm{E}(N) = \frac{1/2}{\left(1 - \frac{1}{2}\right)^2} = 2.$$

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3. *Suppose X is a random variable with* $\mathrm{Var}(X) = 0$. *Compute the mass function of X.*

   Solution. Recall that

$$\mathrm{Var}(X) = \mathrm{E}\left[(X - \mathrm{E}X)^2\right] = \sum_x (x - \mathrm{E}X)^2 f(x),$$

   where f denotes the mass function of X. Because X has zero variance, $f(x) > 0 \;\Rightarrow\; (x - \mathrm{E}X)^2 = 0$. Equivalently, $\mathrm{P}\{X = \mathrm{E}X\} = 1$, whence it follows that X is nonrandom.

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4. *An urn contains n balls, numbered 1 through n. We pick two balls at random, without replacement, all balls equally likely. The number of the first drawn ball is denoted by X, and the number for the second drawn ball by Y. Compute* $\mathrm{cov}(X, Y)$.

   Solution. Let us first compute the joint mass function f of (X, Y):

$$f(x\,,y) = \begin{cases} \dfrac{1}{n(n-1)} & \text{if } 1 \le x \ne y \le n, \\ 0 & \text{otherwise.} \end{cases}$$

   Therefore, for all $x = 1, \ldots, n$,

$$f_X(x) = \sum_{\substack{y=1 \\ y \ne x}}^{n} f(x\,,y) = \frac{1}{n}.$$

   For other values of x, $f_X(x) = 0$. That is, X is distributed uniformly on $\{1\,,\ldots,n\}$. Similarly, Y is distributed uniformly on $\{1\,,\ldots,n\}$. Therefore,

$$\mathrm{E}X = \mathrm{E}Y = \sum_{k=1}^{n} \frac{k}{n} = \frac{n+1}{2}.$$

Also,

$$\begin{aligned}
\mathrm{E}(XY) &= \sum_{\substack{1\le i\neq j\le n\\ i\neq j}} \frac{ij}{n(n-1)} = \sum_{1\le i,j\le n} \frac{ij}{n(n-1)} - \sum_{\substack{1\le i=j\le n\\ i=j}} \frac{ij}{n(n-1)} \\
&= \frac{1}{n(n-1)} \left\{ \left(\sum_{i=1}^{n} i\right) \left(\sum_{j=1}^{n} j\right) - \sum_{i=1}^{n} i^2 \right\} \\
&= \frac{1}{n(n-1)} \left\{ \left(\frac{n(n+1)}{2}\right)^2 - \sum_{i=1}^{n} i^2 \right\} \\
&= \frac{1}{n(n-1)} \left\{ n^2 \left(\frac{n+1}{2}\right)^2 - \sum_{i=1}^{n} i^2 \right\} \\
&= \frac{n}{n-1} \left(\frac{n+1}{2}\right)^2 - \frac{1}{n(n-1)} \sum_{i=1}^{n} i^2 \\
&= \frac{n}{n-1} \left(\frac{n+1}{2}\right)^2 - \frac{1}{n(n-1)} \frac{n(2n^2+3n+1)}{6} \\
&= \frac{n}{n-1} \left(\frac{n+1}{2}\right)^2 - \frac{1}{n-1} \frac{2n^2+3n+1}{6}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathrm{cov}(X,Y) &= \frac{n}{n-1} \left(\frac{n+1}{2}\right)^2 - \frac{1}{n-1} \frac{2n^2+3n+1}{6} \\
&= \frac{1}{n-1} \left(\frac{n+1}{2}\right)^2 - \frac{1}{n-1} \frac{2n^2+3n+1}{6} \\
&= \frac{n^2+2n+1}{4(n-1)} - \frac{2n^2+3n+1}{6(n-1)} \\
&= \frac{1}{12(n-1)} \left[\left(3n^2+6n+3\right) - \left(4n^2+6n+2\right) \right] \\
&= -\frac{n^2-1}{12(n-1)} = -\frac{(n-1)(n+1)}{12(n-1)} \\
&= -\frac{n+1}{12}.
\end{aligned}$$