Solutions to Midterm 1

Math 5010-1, Spring 2007, University of Utah

---

1. *In* 2005*, the United Nations reported that approximately* 50.74 *percent of the adults in the US are women. Suppose we take a random sample of* 100 *people, without replacement, from this population. What are the odds that there are between* 50 *and* 51 *women in this sample?*

   Solution: Let N denote the population of the United States. Define W to be the total number of women, so that $W \approx 0.5074N$. Then the probability that there are 50 or 51 women is

   $$\frac{\binom{W}{50}\binom{N-W}{50}}{\binom{N}{100}} + \frac{\binom{W}{51}\binom{N-W}{49}}{\binom{N}{100}}.$$

   There is a good approximation to this. Since N is very large compared to the sample size of 100, sampling with replacement is pretty much the same as sampling without replacement. Therefore, the answer is close to the binomial probabilities,

   $$\binom{100}{50}0.5074^{50} \times 0.4926^{50} + \binom{100}{51}0.5074^{51} \times 0.4926^{49} \approx 0.158.$$

   There is a sense that this can be proved, but that is another matter.

---

2. *There are k coins on the table. Coin i tosses heads with probability i/k for every $i = 1, \ldots, k$. You choose one at random—all are equally likely—and toss it n times independently. It turns up heads. What is the probability that you had chosen coin k?*

   Solution: Let C_i denote the event that the ith coin is selected. Let H_n denote the event that the first tosses are all heads. By independence, $\mathrm{P}(H_n \,|\, C_i) = (i/k)^n$. Therefore, we can apply the Bayes theorem to find that

   $$\mathrm{P}(C_k \,|\, H_n) = \frac{\mathrm{P}(H_n \,|\, C_k)\mathrm{P}(C_k)}{\sum_{i=1}^k \mathrm{P}(H_n \,|\, C_i)\mathrm{P}(C_i)} = \frac{\frac{1}{k}}{\sum_{i=1}^k \left(\frac{i}{k}\right)^n \left(\frac{1}{k}\right)} = \frac{1}{\sum_{i=1}^k \left(\frac{i}{k}\right)^n}.$$

   If you do this for $n = 1$, then that is acceptable. In that case, things simplify because $\sum_{i=1}^k (i/k) = \frac{1}{2}(k+1)$. Therefore, $\mathrm{P}(C_k \,|\, H_1) = 2/(k+1)$ in that case.

---

3. *There are n light bulbs in a storage facility, k of which are not functional. You select r at random, and without replacement. What is the probability that you select ℓ non-functional bulb? You may assume that all light bulbs are equally likely to be selected.*

   Solution: The probability is

$$\frac{\binom{k}{l}\binom{n-k}{r-l}}{\binom{n}{r}},$$

   assuming, of course, that k, l, n, r are integers that satisfy: (i) $0 \leq l \leq k$; (ii) $0 \leq r - l \leq n - k$; and (iii) $0 \leq r \leq n$. Otherwise, the probability is zero.

---

4. *A fair die is cast 112 times. What is the probability that we roll two dots at least twice?*

   Solution: I will state things in the language of binomial random variables because it will hopefully make things more clear. [Words to the wise: Binomials are not really needed here, though.]

   Call it a success every time you roll two dots. The question asks to find $\mathrm{P}\{X \geq 2\}$, where X is the number of successes. Clearly, $X = \mathrm{Bin}(112\,, 1/6)$. Therefore,

$$\begin{aligned}\mathrm{P}\{X \geq 2\} &= \sum_{j=2}^{112} \binom{112}{j} \left(\frac{1}{6}\right)^j \left(\frac{5}{6}\right)^{112-j} \\ &= 1 - \left[\binom{112}{0} \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^{112-0} + \binom{112}{1} \left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^{112-1}\right].\end{aligned}$$

   This can be simplified. Please find better things to do with your time though.

---

5. *A fair coin is tossed, independently, countably many times.*

   (a) *Carefully write down a sample space Ω for this experiment.*

   Solution: All possible infinite sequences of heads and tails.

   (b) *Prove, carefully using the laws of probability, that every element of Ω is possible, but has probability zero.*

   Solution: "Possible" is easy, since you can write any sequence in Ω down inductively. Let S_n denote the first n steps of the sequence. By independence, $\mathrm{P}(S_n) = (1/2)^n$. Note that $S_n \subseteq S_{n-1}$ for all $n \geq 2$. Therefore, $0 = \lim_{n\to\infty} \mathrm{P}(S_n)$ is the same as the probability of $\cap_{n=1}^{\infty} S_n$, which is the event that the entire sequence appears.