## Chapter 7 Problems

**2.** This is simply because $F(x) - F(x-) = \mathrm{P}\{X = x\}$.

**3.** Note that

$$\begin{aligned}\mathrm{E}X &= \int_0^1 x f(x)\,dx \\ &= \frac{1}{B(a\,,b)} \int_0^1 x^{a+1-1}(1-x)^{b-1}\,dx \\ &= \frac{B(a+1\,,b)}{B(a\,,b)}.\end{aligned}$$

Similarly,

$$\begin{aligned}\mathrm{E}(X^2) &= \int_0^1 x^2 f(x)\,dx \\ &= \frac{1}{B(a\,,b)} \int_0^1 x^{a+2-1}(1-x)^{b-1}\,dx \\ &= \frac{B(a+2\,,b)}{B(a\,,b)}.\end{aligned}$$

Therefore,

$$\mathrm{Var}(X) = \frac{B(a+2\,,b)}{B(a\,,b)} - \left(\frac{B(a+1\,,b)}{B(a\,,b)}\right)^2.$$

**6.** We compute directly: For all real numbers $a$,

$$\begin{aligned}F(a) &= \int_{-\infty}^a \exp\left(-x - e^{-x}\right)\,dx \\ &= \int_{-a}^\infty \exp\left(y - e^y\right)\,dy \qquad [y = -x] \\ &= \int_{e^{-a}}^\infty \exp\left(\ln z - z\right)\,\frac{dz}{z} \qquad [z = e^y; y = \ln z; dy = dz/z] \\ &= \int_{e^{-a}}^\infty e^{-z}\,dz \qquad [\text{since } \exp(\ln z) = z] \\ &= \exp\left(-e^{-a}\right).\end{aligned}$$

**7.** Here, $f(x) = \lambda e^{-\lambda x}$ if $x \geq 0$; else, $f(x) = 0$. Let us first compute the density of $Y = e^{aX}$ in the case that $a > 0$. In that case, for all $y > 1$,

$$\begin{aligned}F_Y(y) &= \mathrm{P}\left\{e^{aX} \leq y\right\} \\ &= \mathrm{P}\left\{X \leq \frac{1}{a}\ln y\right\} = F_X\left(\frac{1}{a}\ln y\right).\end{aligned}$$

Therefore, whenever $y > 1$,

$$f_Y(y) = \frac{1}{ay} f_X\left(\frac{1}{a}\ln y\right) = \frac{\lambda}{ay}\exp\left(-\frac{\lambda}{a}\ln y\right) = \frac{\lambda}{ay} y^{-\lambda/a} = \frac{\lambda}{a} y^{-(\lambda/a)-1}.$$

Else, $f_Y(y) = 0$ if $y \le 1$. Consequently,

$$\mathrm{E}Y = \int_1^\infty y\frac{\lambda}{a} y^{-(\lambda/a)-1}\,dy = \frac{\lambda}{a}\int_1^\infty \frac{1}{y^{\lambda/a}}\,dy = \begin{cases} \dfrac{\lambda/a}{1+(\lambda/a)} & \text{if } \lambda > a, \\ \infty & \text{otherwise.} \end{cases}$$

If, on the other hand, $a < 0$, then $aX \ge 0$, and this means that $0 \le Y \le 1$ with probability one. That is, for all $0 < y < 1$,

$$\begin{aligned} F_Y(y) &= \mathrm{P}\left\{e^{aX} \le y\right\} \\ &= \mathrm{P}\left\{X \ge \frac{1}{a}\ln y\right\} = 1 - F_X\left(\frac{1}{a}\ln y\right). \end{aligned}$$

Therefore,

$$f_Y(y) = \begin{cases} -\dfrac{\lambda}{a} y^{-(\lambda/a)-1} & \text{if } 0 < y < 1, \\ 0 & \text{otherwise.} \end{cases}$$

In this case,

$$\mathrm{E}Y = \frac{\lambda}{a}\int_0^1 y^{-\lambda/a}\,dy = \begin{cases} \dfrac{\lambda/a}{1-(\lambda/a)} & \text{if } 0 < \lambda < a, \\ \infty & \text{otherwise.} \end{cases}$$

**15.** For all $c$,

$$\mathrm{E}\left[(X-c)^2\right] = \mathrm{E}(X^2) - 2c\mathrm{E}X + c^2,$$

provided that $\mathrm{E}(X^2) < \infty$. Call this expression $f(c)$ and note that

$$f'(c) = -2\mathrm{E}X + 2c \qquad f''(c) = 2.$$

This implies that $f$ is minimized at $c_{\min} = \mathrm{E}X$. Also,

$$f\left(c_{\min}\right) = \mathrm{E}\left[(X - \mathrm{E}X)^2\right] = \mathrm{Var}(X).$$

## Chapter 8 Problems

**4.** We know that for all $a \ge 0$,

$$F_{X+Y}(a) = \int_0^a \int_0^{a-y} g(x+y)\,dx\,dy.$$

Note that for all $y$ fixed, the inner integral is equal to:

$$\int_0^{a-y} g(x+y)\,dx = \int_y^a g(z)\,dz. \qquad [z = x+y]$$

Therefore,

$$F_{X+Y}(a) = \int_0^a \int_y^a g(z)\,dz\,dy = \int_0^a \int_0^z dy\, g(z)\,dz = \int_0^a zg(z)\,dz.$$

by reversing the order of the two integrals. Consequently,

$$f_{X+Y}(z) = \begin{cases} zg(z) & \text{if } z \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

**5(b).** We know from our previous assignment that $c = 1/(2\pi)$, and so

$$f(x\,,y) = \frac{1}{2\pi\,(1+x^2+y^2)^{3/2}}.$$

Integrate $[dy]$ to find that

$$\begin{aligned} f_X(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{(1+x^2+y^2)^{3/2}}\,dy \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{(1+x^2+(1+x^2)z^2)^{3/2}}\sqrt{1+x^2}\,dz \qquad [z = y/\sqrt{1+x^2}] \\ &= \frac{1}{2\pi(1+x^2)}\int_{-\infty}^{\infty} \frac{dz}{(1+z^2)^{3/2}} = \frac{C}{1+x^2}, \end{aligned}$$

for some $C$ which does not depend on $x$. It follows then that $X$ is Cauchy, and $C = 1/\pi$ (why?).

**6.** First, note that

$$z^2x^2 = w - x^2 \quad \Rightarrow \quad x^2 = \frac{w}{1+z^2}.$$

Here, we are interested over the range $x, y \geq 0$. Therefore,

$$x = \sqrt{\frac{w}{1+z^2}} \quad \text{and} \quad y = xz = z\sqrt{\frac{w}{1+z^2}}.$$

Now,

$$\frac{\partial x}{\partial w} = \frac{1}{\partial w/\partial x} = \frac{1}{2x},$$

Also,

$$\frac{\partial x}{\partial z} = \frac{1}{\partial z/\partial x} = -\frac{x^2}{y}.$$

Similarly,

$$\frac{\partial y}{\partial w} = \frac{1}{\partial w/\partial y} = \frac{1}{2y},$$

and

$$\frac{\partial y}{\partial z} = \frac{1}{\partial z/\partial y} = \frac{1}{x}.$$

Therefore,

$$\begin{aligned}
J(w\,,z) &= \frac{\partial x}{\partial w}\frac{\partial y}{\partial z} - \frac{\partial x}{\partial z}\frac{\partial y}{\partial w} \\
&= \left(\frac{1}{2x} \times \frac{1}{x}\right) - \left(-\frac{x^2}{y} \times \frac{1}{2y}\right) \\
&= \frac{1}{2x^2} + \frac{x^2}{2y^2} = \frac{x^2+y^2}{2x^2y^2}.
\end{aligned}$$

Solve for $(w\,,z)$ instead of $(x\,,y)$: $x^2 + y^2 = w^2$; and

$$x^2y^2 = \frac{w}{1+z^2} \times \frac{z^2w}{1+z^2} = \frac{wz^2}{(1+z^2)^2}.$$

Therefore,

$$J(w\,,z) = \frac{w^2}{2(1+z^2)^2/wz^2} = \frac{1}{2}w^3\frac{z^2}{(1+z^2)^2}.$$

Now plug to find that

$$\begin{aligned}
f_{W,Z}(w\,,z) &= \frac{1}{2}w^3\frac{z^2}{(1+z^2)^2}f_{X,Y}(x\,,y) \\
&= \frac{1}{2}w^3\frac{z^2}{(1+z^2)^2}\frac{1}{2\pi\left(1+x^2+y^2\right)^{3/2}} \\
&= \frac{1}{4\pi}\frac{z^2}{(1+z^2)^2}\frac{w^3}{\left(1+w^2\right)^{3/2}}.
\end{aligned}$$

In particular, $W$ and $Z$ are independent, since $f_{W,Z}(w\,,z)$ is the product of a function of $w$ and a function of $z$; why does this do the job?

**10.** Solve for $x$ and $y$ to obtain:

$$x = \frac{u+v}{2}, \quad y = \frac{u-v}{2}.$$

Therefore,

$$J(u\,,v) = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = -\frac{1}{2},$$

and therefore,

$$f_{U,V}(u\,,v) = \frac{1}{2}f_{X,Y}\left(\frac{u+v}{2}\,,\,\frac{u-v}{2}\right).$$

If $X = N(0\,,\sigma^2)$ and $Y = N(0\,,\tau^2)$, then

$$f_{X,Y}(x\,,y) = \frac{1}{2\pi\sigma\tau}\exp\left(-\left[\frac{x^2}{2\sigma^2}\right] - \left[\frac{y^2}{2\tau^2}\right]\right).$$

And hence,

$$f_{U,V}(u\,,v) = \frac{1}{4\pi\sigma\tau}\exp\left(-\left[\frac{(u+v)^2}{8\sigma^2}\right] - \left[\frac{(u-v)^2}{8\tau^2}\right]\right).$$

If $\tau \neq \sigma$, then this cannot be written as a function of $u$ times a function of $v$. Therefore, $U$ and $V$ are not independent.

Else, if $\tau = \sigma$, then

$$\begin{aligned} f_{U,V}(u\,,v) &= \frac{1}{4\pi\sigma^2}\exp\left(-\frac{1}{8\sigma^2}\left[(u+v)^2 + (u-v)^2\right]\right) \\ &= \frac{1}{4\pi\sigma^2}e^{-u^2/4\sigma^2}e^{-v^2/4\sigma^2}. \end{aligned}$$

This has the correct product form, so $U$ and $V$ are independent in this case.