Chapter 7 Problems

32. We know that the density function f_α of X_α is

$$f_\alpha(x) = \begin{cases} \dfrac{1}{(\alpha-1)!} x^{\alpha-1} e^{-x} & \text{if } x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

We want to find the density function ϕ_α of $Y_\alpha = (X_\alpha - \alpha)/\sqrt{\alpha}$. First we find F_{Y_α}. Then differentiate to obtain ϕ_α. Of course,

$$F_{Y_\alpha}(x) = \Pr\left\{X_\alpha \leq x\sqrt{\alpha} + \alpha\right\} = F_{X_\alpha}\left(x\sqrt{\alpha} + \alpha\right).$$

First of all, if $x\sqrt{\alpha} + \alpha < 0$, then $F_{Y_\alpha}(x) = 0$. The condition on x is equivalent to $x\sqrt{\alpha} < -\alpha$, which is also equivalent to $x < -\sqrt{\alpha}$. On the other hand, if $x \geq -\sqrt{\alpha}$, then

$$\begin{aligned} f_{Y_\alpha}(x) &= \frac{d}{dx} F_{X_\alpha}\left(x\sqrt{\alpha} + \alpha\right) \times \sqrt{\alpha} \\ &= \sqrt{\alpha} \times f_{X_\alpha}\left(x\sqrt{\alpha} + \alpha\right) \\ &= \frac{\sqrt{\alpha}}{(\alpha-1)!}\left(x\sqrt{\alpha} + \alpha\right)^{\alpha-1} e^{-x\sqrt{\alpha}-\alpha}. \end{aligned}$$

Recall that $(\alpha - 1)! = \Gamma(\alpha)$. Using this, we can apply Stirling's formula (§7.19, p. 332, eq. (3)) to find that as $\alpha \to \infty$,

$$(\alpha - 1)! \approx e^{-\alpha}\alpha^{\alpha-\frac{1}{2}}\sqrt{2\pi}.$$

Therefore,

$$\begin{aligned} f_{Y_\alpha}(x) &\approx \frac{\sqrt{\alpha}}{e^{-\alpha}\alpha^{\alpha-\frac{1}{2}}\sqrt{2\pi}}\left(x\sqrt{\alpha} + \alpha\right)^{\alpha-1} e^{-x\sqrt{\alpha}-\alpha} \\ &= \frac{1}{\alpha^{\alpha-1}\sqrt{2\pi}}\left(x\sqrt{\alpha} + \alpha\right)^{\alpha-1} e^{-x\sqrt{\alpha}} \end{aligned}$$

As $\alpha \nearrow \infty$,

$$\begin{aligned} \left(x\sqrt{\alpha} + \alpha\right)^{\alpha-1} &= \frac{\left(x\sqrt{\alpha} + \alpha\right)^{\alpha}}{x\sqrt{\alpha} + \alpha} \\ &\approx \frac{\left(x\sqrt{\alpha} + \alpha\right)^{\alpha}}{\alpha} \\ &= \frac{\alpha^\alpha\left(\dfrac{x}{\sqrt{\alpha}} + 1\right)^{\alpha}}{\alpha} \\ &= \alpha^{\alpha-1}\left(\frac{x}{\sqrt{\alpha}} + 1\right)^{\alpha}. \end{aligned}$$

Apply Taylor expansion to find that as $\alpha \nearrow \infty$,

$$\begin{aligned}
\ln\left[\left(\frac{x}{\sqrt{\alpha}}+1\right)^{\alpha}\right] &= \alpha \ln\left(\frac{x}{\sqrt{\alpha}}+1\right) \\
&= \alpha\left(\frac{x}{\sqrt{\alpha}}-\frac{x^2}{2\alpha}+\text{small terms}\right) \\
&= x\sqrt{\alpha}-\frac{x^2}{2}+\text{small terms.}
\end{aligned}$$

That is,

$$\left(\frac{x}{\sqrt{\alpha}}+1\right)^{\alpha} \approx \exp\left(x\sqrt{\alpha}-\frac{x^2}{2}\right).$$

Plug to find that

$$f_{Y_\alpha}(x) \approx \frac{e^{-x^2/2}}{\sqrt{2\pi}} = \phi(x).$$

Chapter 8 Problems

3. (a) To find c compute:

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x\,,y)\,dx\,dy = c\int_0^{\infty}\int_0^{\infty} e^{-x-y}\,dx\,dy \\
&= c\left(\int_0^{\infty} e^{-x}\,dx\right)\left(\int_0^{\infty} e^{-y}\,dy\right) = c.
\end{aligned}$$

Therefore, $c=1$.

(b) In order to compute $\Pr\{X+Y>1\}$, first find the region of integration and plot it. You will find that

$$\begin{aligned}
\Pr\{X+Y>1\} &= 1-\Pr\{X+Y\le 1\} \\
&= 1-\int_0^1\int_{1-x}^1 e^{-x-y}\,dy\,dx = 1-\int_0^1\left(\int_{1-x}^1 e^{-y}\,dy\right)e^{-x}\,dx \\
&= 1-\int_0^1\left(e^{1-x}-e^{-x}\right)e^{-x}\,dx = 1-(e-1)\int_0^1 e^{-2x}\,dx \\
&= 1-\frac{e-1}{2}\left(1-e^{-2}\right).
\end{aligned}$$

(c) $\Pr\{X<Y\}=\Pr\{X>Y\}$, by symmetry [draw the region of integration! This fact is special to this problem.] Because

$$1=\Pr\{X<Y\}+\Pr\{X>Y\}+\Pr\{X=Y\}=2\Pr\{X<Y\},$$

we have $\Pr\{X<Y\}=\Pr\{X>Y\}=1/2$. Or you can do this by direct integration.

5(a) We know that

$$f(x\,, y) = \frac{c}{\left(1+x^2+y^2\right)^{3/2}}.$$

Therefore,

$$\begin{aligned}
1 &= c\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\left(1+x^2+y^2\right)^{3/2}}\,dx\,dy \\
&= c\int_0^{2\pi}\int_0^{\infty} \frac{1}{\left(1+r^2\right)^{3/2}}\,r\,dr\,d\theta \\
&= c\int_0^{\infty}\left(\int_0^{2\pi} d\theta\right)\frac{1}{\left(1+r^2\right)^{3/2}}\,r\,dr \\
&= 2\pi c\int_0^{\infty} \frac{r}{\left(1+r^2\right)^{3/2}}\,dr.
\end{aligned}$$

Change variables ($w = 1 + r^2$) to find that

$$1 = \pi c\int_1^{\infty}\frac{dw}{w^{3/2}} = 2\pi c.$$

So $c = 1/(2\pi)$.