## Chapter 6 Problems

**2. (a)** For all real numbers $s \neq 1$,

$$G(s) = \frac{1}{n}\sum_{k=1}^{n} s^k = \frac{1}{n}\left[\sum_{k=0}^{\infty} s^k - 1 - \sum_{k=n+1}^{\infty} s^k\right]$$
$$= \frac{1}{n}\left[\frac{1}{1-s} - 1 - \frac{s^{n+1}}{1-s}\right] = \frac{s - s^{n+1}}{n(1-s)}.$$

If $s = 1$, then $G(s) = 1$, trivially.

**(b)** For all real numbers $s \neq 1$,

$$G(s) = \frac{1}{2n+1}\sum_{k=-n}^{n} s^k = \frac{s^{-n}}{2n+1}\sum_{k=-n}^{n} s^{k+n} = \frac{s^{-n}}{2n+1}\sum_{j=0}^{2n} s^j$$
$$= \frac{s^{-n}}{2n+1}\left[\frac{1-s^{2n+1}}{1-s}\right] = \frac{s^{-n} - s^{n+1}}{(2n+1)(1-s)}.$$

If $s = 1$, then $G(s) = 1$, trivially.

**(c)** We write, for all $k \geq 1$,

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}.$$

Therefore, as long as $|s| < 1$,

$$G(s) = \sum_{k=1}^{\infty} \frac{s^k}{k} - \sum_{k=1}^{\infty} \frac{s^k}{k+1}.$$

Now,

$$\sum_{k=1}^{\infty} \frac{s^k}{k} = \sum_{k=1}^{\infty} \int_0^s x^{k-1}\, dx = \int_0^s \left(\sum_{k=1}^{\infty} x^{k-1}\right) dx$$
$$= \int_0^s \frac{1}{1-x}\, dx = \ln\left(\frac{1}{1-s}\right).$$

Also,

$$\sum_{k=1}^{\infty} \frac{s^k}{k+1} = \frac{1}{s}\sum_{k=1}^{\infty} \frac{s^{k+1}}{k+1} = \frac{1}{s}\sum_{j=2}^{\infty} \frac{s^j}{j}$$
$$= \frac{1}{s}\left[\sum_{j=1}^{\infty} \frac{s^j}{j} - s\right]$$
$$= \frac{1}{s}\left[\ln\left(\frac{1}{1-s}\right) - s\right]$$
$$= \frac{1}{s}\ln\left(\frac{1}{1-s}\right) - 1.$$

Therefore, whenever $|s| < 1$,

$$G(s) = 1 + \left[1 - \frac{1}{s}\right] \ln\left(\frac{1}{1-s}\right).$$

**(d)** If $G$ converges [absolutely], then we can write it

$$G(s) = \sum_{k=1}^{\infty} \frac{s^k}{2k(k+1)} + \sum_{k=-\infty}^{-1} \frac{s^k}{2k(k-1)}.$$

The first sum converges if $|s| \leq 1$, whereas the second if $|s| \geq 1$. Therefore, the only convergent values are for $s = \pm 1$. Moreover, $G(1) = 1$ because after a change of variables,

$$\begin{aligned} G(1) = \sum_{k=1}^{\infty} \frac{1}{k(k+1)} &= \lim_{N\to\infty} \sum_{k=1}^{N} \frac{1}{k(k+1)} \\ &= \lim_{N\to\infty} \sum_{k=1}^{N} \left(\frac{1}{k} - \frac{1}{k+1}\right) = \lim_{N\to\infty} \left(1 - \frac{1}{1+N}\right) = 1. \end{aligned}$$

**(e)** Clearly,

$$\begin{aligned} G(s) &= \frac{1-c}{1+c} \sum_{k=-\infty}^{\infty} s^k c^{|k|} \\ &= \frac{1-c}{1+c} \sum_{k=0}^{\infty} (sc)^k + \frac{1-c}{1+c} \sum_{k=-\infty}^{-1} \left(\frac{c}{s}\right)^{-k} \\ &= \frac{1-c}{1+c} \sum_{k=0}^{\infty} (sc)^k + \frac{1-c}{1+c} \sum_{j=1}^{\infty} \left(\frac{c}{s}\right)^{j}. \end{aligned}$$

The first sum converges as long as $|s| \leq 1$. But the second converges only if $|s| > c$. Therefore, as long as $c < |s| \leq 1$,

$$\begin{aligned} G(s) &= \frac{1-c}{1+c} \times \frac{1}{1-sc} + \frac{1-c}{1+c} \left[\sum_{j=0}^{\infty} \left(\frac{c}{s}\right)^{j} - 1\right] \\ &= \frac{1-c}{1+c} \times \frac{1}{1-sc} + \frac{1-c}{1+c} \left[\frac{1}{1-(c/s)} - 1\right]. \end{aligned}$$

**17.** According to Example (6) on page 245,

$$\begin{aligned} G(s) &= \left(\frac{1-(\lambda/n)}{1-\lambda s/n}\right)^n \\ &= \left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda s}{n}\right)^{-n} \\ &\to \frac{e^{-\lambda}}{e^{-\lambda s}} = e^{-\lambda(1-s)}. \end{aligned}$$

This proves that the mass function of $X_n$ converges to that of Poisson($\lambda$) as $n \to \infty$. Also,

$$\begin{aligned} G(s) &= \left(\frac{p}{1-qs}\right)^n = p^n(1-qs)^{-n} \\ G'(s) &= nqp^n(1-qs)^{-n-1} \\ G''(s) &= n(n+1)q^2p^n(1-qs)^{-n-2}. \end{aligned}$$

Therefore,

$$\begin{aligned} G'(1) &= nqp^n(1-q)^{-n-1} = nqp^np^{-n-1} = \frac{nq}{p} \\ G''(1) &= n(n+1)q^2p^np^{-n-2} = \frac{n(n+1)q^2}{p^2}. \end{aligned}$$

Consequently, $\mathrm{E}(X) = nq/p$, and

$$\begin{aligned} \mathrm{Var}(X) &= G''(1) + G'(1) - [G'(1)]^2 \qquad \text{(p. 237)} \\ &= \frac{n(n+1)q^2}{p^2} + \frac{nq}{p} - \left(\frac{nq}{p}\right)^2 \\ &= \frac{nq^2}{p^2} + \frac{nq}{p} \\ &= \frac{nq}{p}\left[\frac{q}{p}+1\right] = \frac{nq}{p^2}. \end{aligned}$$

## Chapter 7 Problems

**1.** Because $(x-\alpha)(\beta-x) = -x^2 + (\beta-\alpha)x - \alpha\beta$, $f(x)$ has to be zero unless $x$ lies between $\alpha$ and $\beta$. If $\alpha = \beta$, then this cannot be done. If $\alpha < \beta$, then we choose $c(\alpha\,,\beta)$ so that

$$\begin{aligned} \frac{1}{c(\alpha,\beta)} &= \int_\alpha^\beta \left[-x^2 + (\beta-\alpha)x - \alpha\beta\right]\, dx \\ &= -\frac{\beta^3-\alpha^3}{3} + \frac{(\beta-\alpha)(\beta^2-\alpha^2)}{2} - \alpha\beta(\beta-\alpha). \end{aligned}$$

Else, if $\alpha > \beta$, then $c(\alpha\,,\beta)$ is minus one times the reciprocal of the preceding term.

**9.** Compute directly to find that

$$\begin{aligned}
\mathrm{P}\left(X > x + \frac{a}{x} \;\middle|\; X > x\right) &= \frac{\mathrm{P}\{X > x + (a/x)\}}{\mathrm{P}\{X > x\}} \\
&= \frac{\int_{x+(a/x)} e^{-y^2/2}\,dy}{\int_x^\infty e^{-y^2/2}\,dy} \\
&\sim \frac{\left(\int_{x+(a/x)} e^{-y^2/2}\,dy\right)'}{\left(\int_x^\infty e^{-y^2/2}\,dy\right)'},
\end{aligned}$$

where "prime" denotes $d/dx$. We apply the fundamental theorem of calculus to find that

$$\left(\int_x^\infty e^{-y^2/2}\,dy\right)' = -\exp\left(-\frac{x^2}{2}\right).$$

Also, by the fundamental theorem of calculus and the change rule,

$$\begin{aligned}
\left(\int_{x+(a/x)}^\infty e^{-y^2/2}\,dy\right)' &= -\exp\left(-\frac{(x+(a/x))^2}{2}\right) \times \left(x + \frac{a}{x}\right)' \\
&= -\exp\left(-\frac{(x+(a/x))^2}{2}\right) \times \left(1 - \frac{a}{x^2}\right) \\
&\sim -\exp\left(-\frac{(x+(a/x))^2}{2}\right) \\
&= \exp\left(-\frac{x^2+2a+(a^2/x^2)}{2}\right) \sim \exp\left(-\frac{x^2+2a}{2}\right).
\end{aligned}$$

The claim of the problem follows from these computations.

**10.** (a) $F_{|X|}(a) = \mathrm{P}\{|X| \le a\} = \mathrm{P}\{-a \le X \le a\} = \Phi(a) - \Phi(-a)$, provided that $a \ge 0$. Else, $F_{|X|}(a) = 0$. By symmetry, $\Phi(a) = 1 - \Phi(-a)$, whence follows the claim.

(b) As before, $F_{|X|}(a) = F(a) - F(-a)$. Differentiate to find that $f_{|X|}(a) = f(a) + f(-a)$, when the density $f$ of $X$ exists.

**26.** For any integer $k \ge 1$, $X \ge k$ if and only $\log U / \log(1-p) \ge k-1$. Therefore,

$$\begin{aligned}
\mathrm{P}\{X \ge k\} &= \mathrm{P}\left\{\frac{\log U}{\log q} \ge k-1\right\} \\
&= \mathrm{P}\left\{\log U \le (k-1)\log q\right\} \qquad \text{(because } \log q \le 0\text{)} \\
&= \mathrm{P}\left\{U \le q^{k-1}\right\} = \int_0^{q^{k-1}} dy = q^{k-1}.
\end{aligned}$$

Therefore, $F(n) = 1 - \mathrm{P}\{X \ge n+1\} = 1 - q^n$ for all $n \ge 0$. It follows that $f(n) = \mathrm{P}\{X = n\} = F(n) - F(n-1) = q^{n-1} - q^n = q^{n-1}(1-q) = pq^{n-1}$ for all $n \ge 0$. This proves that $X$ is geometric($p$).