Chapter 5 Problems

1. See your lecture notes.

2.

(a) $\Pr\{X > Y\} = 0$.

(b) $\Pr\{X \geq Y\} = f(2\,,2) = 1/16$.

(c) $\Pr\{X + Y \text{ is odd}\} = f(1\,,2) + f(1\,,4) + f(2\,,3) = 1/2$.

(d) $\Pr\{X - Y \leq 1\} = 1 - \Pr\{X > Y + 1\} = 1$.

4. We know that

$$0 = \mathrm{E}\left[(X - Y)^2\right] = \sum_{x,y} (x - y)^2 f(x\,,y).$$

Therefore, for all $x \neq y$, $f(x\,,y)$ must be zero. In particular,

$$\Pr\{X = Y\} = \sum_x f(x\,,x) = \sum_{x,y} f(x\,,y) = 1.$$

8. First consider the case that the initial symbol is a zero. In order to transmit it correctly, you either pass it through correctly both times, or transmit it incorrectly at both channels. That is,

$$\Pr\{\text{correct transmission} \,|\, \text{symbol is zero}\} = p^2 + (1-p)^2 = 1 - 2p + 2p^2.$$

Similarly,

$$\Pr\{\text{correct transmission} \,|\, \text{symbol is one}\} = 1 - 2p + 2p^2.$$

Therefore, apply Bayes's rule to discover that

$$\Pr\{\text{correct transmission}\} = (1-2p+2p^2)\frac{1}{2} + (1-2p+2p^2)\frac{1}{2} = 1-2p+2p^2.$$

Define $h(p) = 1 - 2p + 2p^2$ to find that $h'(p) = -2 + 4p$ and $h''(p) = 4$. Therefore, setting $h' = 0$ yields a minimum at $p = 1/2$, as desired.

If we had three independent channels, then similar considerations yield

$$\Pr\{\text{correct transmission}\} = p^3 + 3(1-p)^2 p.$$

10. Let us first compute the joint mass function f of (X, Y):

$$f(x\,,y) = \begin{cases} \dfrac{1}{n(n-1)} & \text{if } 1 \leq x \neq y \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, for all $x = 1, \ldots, n$,

$$f_X(x) = \sum_{\substack{y=1 \\ y \neq x}}^{n} f(x\,, y) = \frac{1}{n}.$$

For other values of x, $f_X(x) = 0$. That is, X is distributed uniformly on $\{1\,, \ldots, n\}$. Similarly, Y is distributed uniformly on $\{1\,, \ldots, n\}$. Therefore,

$$\mathrm{E}X = \mathrm{E}Y = \sum_{k=1}^{n} \frac{k}{n} = \frac{n+1}{2}.$$

Also,

$$\begin{aligned}
\mathrm{E}(XY) &= \sum_{\substack{1 \leq i \neq j \leq n \\ i \neq j}} \frac{ij}{n(n-1)} = \sum_{1 \leq i,j \leq n} \frac{ij}{n(n-1)} - \sum_{\substack{1 \leq i = j \leq n \\ i = j}} \frac{ij}{n(n-1)} \\
&= \frac{1}{n(n-1)} \left\{ \left(\sum_{i=1}^{n} i \right) \left(\sum_{j=1}^{n} j \right) - \sum_{i=1}^{n} i^2 \right\} \\
&= \frac{1}{n(n-1)} \left\{ \left(\frac{n(n+1)}{2} \right)^2 - \sum_{i=1}^{n} i^2 \right\} \\
&= \frac{1}{n(n-1)} \left\{ n^2 \left(\frac{n+1}{2} \right)^2 - \sum_{i=1}^{n} i^2 \right\} \\
&= \frac{n}{n-1} \left(\frac{n+1}{2} \right)^2 - \frac{1}{n(n-1)} \sum_{i=1}^{n} i^2 \\
&= \frac{n}{n-1} \left(\frac{n+1}{2} \right)^2 - \frac{1}{n(n-1)} \frac{n(2n^2+3n+1)}{6} \\
&= \frac{n}{n-1} \left(\frac{n+1}{2} \right)^2 - \frac{1}{n-1} \frac{2n^2+3n+1}{6}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\operatorname{cov}(X,Y) &= \frac{n}{n-1}\left(\frac{n+1}{2}\right)^2 - \frac{1}{n-1}\frac{2n^2+3n+1}{6}\\
&= \frac{1}{n-1}\left(\frac{n+1}{2}\right)^2 - \frac{1}{n-1}\frac{2n^2+3n+1}{6}\\
&= \frac{n^2+2n+1}{4(n-1)} - \frac{2n^2+3n+1}{6(n-1)}\\
&= \frac{1}{12(n-1)}\left[\left(3n^2+6n+3\right) - \left(4n^2+6n+2\right)\right]\\
&= -\frac{n^2-1}{12(n-1)} = -\frac{(n-1)(n+1)}{12(n-1)}\\
&= -\frac{n+1}{12}.
\end{aligned}$$

Therefore, $\operatorname{cov}(X,Y) \to -\infty$ as $n \to \infty$. Next, we note that

$$\mathrm{E}(X^2) = \mathrm{E}(Y^2) = \sum_{k=1}^{n} \frac{k^2}{n} = \frac{2n^2+3n+1}{6}.$$

Therefore,

$$\begin{aligned}
\operatorname{Var}(X) = \operatorname{Var}(Y) &= \frac{2n^2+3n+1}{6} - \left(\frac{n+1}{2}\right)^2\\
&= \frac{2n^2+3n+1}{6} - \frac{n^2+2n+1}{4}\\
&= \frac{n^2}{3} - \frac{n^2}{4} + \text{smaller terms}\\
&= \frac{n^2}{12} + \text{smaller terms}.
\end{aligned}$$

Therefore,

$$\rho(X,Y) = \frac{-(n+1)/12}{(n^2/2) + \text{smaller terms}} \to 0.$$

In fact, you should intuitively think of X and Y as being "almost independent," though it is not clear what this might mean mathematically speaking (I hope!). [This can be made sense of, but needs much more development.]