Chapter 4 Problems

9. By independence,

$$\Pr\{X=k\} = \sum_{j\geq 1:\, k\leq 10^j} \Pr\{X=k\,, N=j\}$$
$$= \sum_{j\geq \log k} \frac{1}{10^j}\Pr\{N=j\}.$$

But N = geometric(1/2). Therefore, $\Pr\{N=j\} = (1/2)(1/2)^{j-1} = (1/2)^j$, whence

$$\Pr\{X=k\} = \sum_{j\geq \log k} \frac{1}{20^j}.$$

Recall that if $0<r<1$ and $n\geq 1$ is an integer, then $\sum_{j\geq n} r^j = r^n/(1-r)$ [Lecture 9]. Apply this with $r=1/20$ and $n=\lceil \log k\rceil$, where $\lceil x\rceil$ denotes the smallest integer $\geq x$. In other words,

$$\Pr\{X=k\} = \frac{(1/20)^{\lceil \log k\rceil}}{1-(1/20)} = \frac{1}{19}\cdot\frac{1}{20^{\lceil \log k\rceil -1}} \quad \text{k=1,2,...} .$$

For other values of k, $\Pr\{X=k\}=0$. Of course,

$$\mathrm{E}(X) = \frac{1}{19}\sum_{k=1}^{\infty}\frac{k}{20^{\lceil \log k\rceil}},$$

but can this be simplified a bit? To see that the answer is "yes," note that $\lceil \log k\rceil \leq \log k + 1$. Therefore,

$$\mathrm{E}(X) \geq \frac{1}{20\times 19}\sum_{k=1}^{\infty}\frac{k}{20^{\log k}} \geq \frac{1}{20\times 19}\sum_{k=1}^{\infty}\frac{k}{k^2} = \infty.$$

This proves that $\mathrm{E}X = \infty$.

11. Compute directly to find that

$$\mathrm{E}\left[(X-a)^2\right] = \mathrm{E}\left[X^2 - 2aX + a^2\right] = \underbrace{\mathrm{E}(X^2) - 2a\mathrm{E}X + a^2}_{g(a)}.$$

Thus, we are interested in minimizing the function g over all a. First of all,

$$g'(a) = -2\mathrm{E}X + 2a.$$

Therefore, $g'(a)=0$ yields $a=\mathrm{E}X$. To see that this is giving us the minimum of g, note that $g''(a) = 2 > 0$. Thus,

$$\min_a g(a) = g(\mathrm{E}X) = \mathrm{E}\left[(X-\mathrm{E}X)^2\right],$$

which is $\mathrm{Var}(X)$.

15. By symmetry,

$$\begin{aligned}\Pr\{X \le a\} &= \sum_{x \le a} f(x) = \sum_{y \le 0} f(y+a) \\ &= \sum_{y \le 0} f(a-y) = \sum_{y \ge 0} f(a+y) \\ &= \sum_{x \ge a} f(x) = \Pr\{X \ge a\}.\end{aligned} \tag{1}$$

Therefore, the median is a. Note that it need not be the case that $\mathrm{P}\{X \le a\} = 1/2$, though (why?).

In order to compute the mean, I first claim that

$$\sum_{x<a} xf(x) + \sum_{x>a} xf(x) = a - af(a). \tag{2}$$

Indeed, a change of variables $[y = x - a]$ reveals that

$$\begin{aligned}\sum_{x<a} xf(x) &= \sum_{y<0} (y+a)f(y+a) \\ &= \sum_{y<0} (y+a)f(a-y) \qquad \text{(by symmetry)} \\ &= \sum_{z>0} (a-z)f(a+z) \qquad (z=-y) \\ &= \sum_{x>a} (2a-x)f(x) \qquad (x=a+z) \\ &= 2a\sum_{x>a} f(x) - \sum_{x>a} xf(x).\end{aligned} \tag{3}$$

By the same argument that led to 1, $\sum_{x>a} f(x) = \sum_{x<a} f(x)$. But

$$1 = \sum_{x<a} f(x) + \sum_{x>a} f(x) + f(a) = 2\sum_{x>a} f(x) + f(a).$$

Therefore,

$$2a\sum_{x>a} f(x) = a(1-f(a)) = a - af(a).$$

Plug this into (3) to obtain (2). Thus,

$$\mathrm{E}X = \sum_{x<a} xf(x) + \sum_{x>a} xf(x) + af(a) = a.$$

18. We have

$$\mathrm{E}\left(\frac{1}{X}\right) = p\sum_{k=1}^{\infty} \frac{1}{k} q^{k-1}.$$

It remains to compute the infinite sum. Note that

$$\frac{1}{k}q^{k-1} = \frac{1}{q} \times \frac{1}{k}q^k = \frac{1}{q}\int_0^q x^{k-1}\,dx.$$

Therefore,

$$\begin{aligned}
\mathrm{E}\left(\frac{1}{X}\right) &= \frac{p}{q}\sum_{k=1}^{\infty}\int_0^q x^{k-1}\,dx = \frac{p}{q}\int_0^q \left(\sum_{k=1}^{\infty} x^{k-1}\right)dx \\
&= \frac{p}{q}\int_0^q \frac{1}{1-x}\,dx = \frac{p}{q}\int_p^1 \frac{1}{y}\,dy \qquad (y = 1-x) \\
&= \frac{p}{q}\left[\ln 1 - \ln p\right] = \frac{p}{q}\ln\left(\frac{1}{p}\right).
\end{aligned}$$

This does the job.

30. Let F_n denote the event that the nth day is fine. We know that $\Pr(F_n \mid F_{n-1}) = p$ and $\Pr(F_n \mid F_{n-1}^c) = p'$. If $u_n = \Pr(F_n)$, then

$$\begin{aligned}
u_n &= \Pr(F_n \mid F_{n-1})\Pr(F_{n-1}) + \Pr(F_n \mid F_{n-1}^c)\Pr(F_{n-1}^c) \\
&= pu_{n-1} + p'(1-u_{n-1}).
\end{aligned}$$

Thus, $u_n = (p-p')u_{n-1} + p'$. Let us first *assume* that

$$u_\infty = \lim_{n\to\infty} u_n \text{ exists.} \tag{4}$$

Then, $u_\infty = (p-p')u_\infty + p'$, and hence

$$u_\infty = \frac{p'}{1-p+p'}.$$

It remains to verify (4). Note that

$$\begin{aligned}
u_n - u_\infty &= (p-p')u_{n-1} + p' - u_\infty \\
&= (p-p')\left(u_{n-1} - u_\infty\right) + p' - u_\infty\left[1-(p-p')\right] \\
&= (p-p')\left(u_{n-1} - u_\infty\right).
\end{aligned}$$

Because this is valid for all $n \geq 2$, we can apply induction to find that

$$\begin{aligned}
(u_n - u_\infty) &= (p-p')(u_{n-1} - u_\infty) \\
&= (p-p')^2(u_{n-2} - u_\infty) = (p-p')^3(u_{n-3} - u_\infty) \\
&= \cdots = (p-p')^j(u_{n-j} - u_\infty) \\
&= \cdots = (p-p')^{n-1}(u_1 - u_\infty).
\end{aligned}$$

Because $|p-p'| < 1$, this proves that $|u_n - u_\infty| \to 0$ as $n \to \infty$, and this proves the existence of the limit.

35. According to Lecture 9 (§2), $Y = k + X$ is negative binomial, and

$$f(x) = \begin{cases} \binom{x-1}{k-1} p^k (1-p)^{x-k} & \text{if } x = k, k+1, k+2, \dots, \\ 0 & \text{otherwise.} \end{cases}$$

This implies **(a)**.

(b) According to Example 10.6 (Lecture 10), $\mathrm{E}(Y) = k/p$, whence $\mathrm{E}(X) = (k/p) - k = kq/p$. We compute $\mathrm{Var}(X) = \mathrm{Var}(Y)$ by first noticing that

$$\begin{aligned} \mathrm{E}[(Y+1)Y] &= \sum_{x=k}^{\infty} (x+1)x \binom{x-1}{k-1} p^k (1-p)^{x-k} \\ &= \sum_{x=k}^{\infty} \frac{(x+1)!}{(k-1)!(x-k)!} p^k q^{x-k} \\ &= \frac{(k+1)k}{p^2} \sum_{x=k}^{\infty} \binom{x+1}{k+1} p^{k+2} q^{(x+2)-(k+2)} \\ &= \frac{(k+1)k}{pq} \sum_{y=k+2}^{\infty} \binom{y-1}{k+1} p^{k+2} q^{y-(k+2)} \\ &= \frac{(k+1)k}{p^2}. \end{aligned}$$

Therefore,

$$\mathrm{E}(Y^2) = \mathrm{E}[(Y+1)Y] - \mathrm{E}Y = \frac{(k+1)k}{p^2} - \frac{k}{p},$$

and hence

$$\mathrm{Var}(Y) = \frac{(k+1)k}{p^2} - \frac{k}{p} - \frac{k^2}{p^2} = \frac{k}{p^2} - \frac{k}{p} = \frac{k}{p}\left(\frac{1}{p} - 1\right) = \frac{kq}{p^2}.$$

Because $\mathrm{Var}(X) = \mathrm{Var}(Y-k) = \mathrm{Var}(Y)$, the preceding also computes the variance of X.