Chapter 3 Problems

1. Let R_j, M_j, and A_j respectively designate the events, {red on jth draw}, {mauve on jth draw}, and {attractive rainbox motif on jth draw}. Because there are 4 red socks, 6 mauve ones, and 8 rainbow-colored socks(!), we have

$$\begin{aligned}\Pr\{\text{match}\} &= \Pr(R_1 \cap R_2) + \Pr(M_1 \cap M_2) + \Pr(A_1 \cap A_2) \\ &= \frac{\binom{4}{2}}{\binom{18}{2}} + \frac{\binom{6}{2}}{\binom{18}{2}} + \frac{\binom{8}{2}}{\binom{18}{2}} = \frac{49}{153}.\end{aligned}$$

5. Worked out during lectures.

27. Let $p = \Pr(H)$, where $H = \{\text{heads}\}$, etc. Note that the probability of the event $A = \{\text{number of heads} = \text{the number of tails}\}$ is the same as the probability that the total number of heads is exactly n. Let $\mathcal{H}$ denote the collection of all arrangements of n heads and n tails [try to digest this first!]. Each $B \in \mathcal{H}$ has probability $p^n(1-p)^n$. Therefore,

$$\Pr(A) = |\mathcal{H}| \cdot p^n(1-p)^n = |\mathcal{H}|(pq)^n,$$

where $|\mathcal{H}|$ denotes the number of elements of $\mathcal{H}$ and $q = 1-p$, as usual. It should now be clear that $|\mathcal{H}|$ is the number of ways we can distribute n heads—and hence also n tails—in $2n$ slots. Thus, $|\mathcal{H}| = \binom{2n}{n}$. Therefore,

$$\Pr(A) = \binom{2n}{n}(pq)^n. \tag{1}$$

This is the desired formula. In order to approximate this expression for large values of n, we need the Stirling formula (p. 96 of your text):

$$n! \sim (n/e)^n\sqrt{2\pi n}, \tag{2}$$

where $a_n \sim b_n$ means $\lim_{n\to\infty}(a_n/b_n) = 1$.

We apply (2) to find that

$$\begin{aligned}\binom{2n}{n} &= \frac{(2n)!}{(n!)^2} \sim \frac{(2n/e)^{2n}\sqrt{4\pi n}}{\left\{(n/e)^n\sqrt{2\pi n}\right\}^2} = \frac{(2n/e)^{2n}\sqrt{4\pi n}}{2\pi n \cdot (n/e)^{2n}} = \frac{2^{2n}}{\sqrt{\pi n}} \\ &= \frac{4^n}{\sqrt{\pi n}}.\end{aligned}$$

Plug this into (1) to find that

$$\Pr(A) \sim \frac{(4pq)^n}{\sqrt{\pi n}} \qquad \text{as } n \to \infty.$$

An interesting feature of this is that when $p = q = 1/2$, then $\Pr(A) \sim 1/\sqrt{\pi n}$ goes to zero as $n \to \infty$, but rather slowly. On the other hand, when $p \neq q$, then $\Pr(A)$ goes to zero exponentially fast as $n \to \infty$. Does this make physical sense to you? [It should if you think about it for a while.]

28. (a) By the binomial theorem,

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = (1 + (-1))^n = 0.$$

(b) This was worked out during the lectures.

(c) Let us write $n = 2m$, since n is even. Then, the method that led to the answer of (b) shows that

$$\begin{aligned}\sum_{k=0}^{m} \binom{2m}{2k} &= \text{ the number of even-sized subsets of } \{1, \ldots, 2m\} \\ &= \frac{1}{2} \times \text{ the total number of subsets of } \{1, \ldots, 2m\} \\ &= \frac{1}{2} \times 2^{2m}.\end{aligned}$$

Clearly, this is the same as 2^{n-1}.

(d) The same as (c), but now we are counting the total number of odd-sized subsets of $\{1, \ldots, n\}$.