## Chapter 2 Problems

**1.** Let $W$ denote the event {windy} and $H$ = {hit target}. We know that: $\Pr(W) = 0.3$; $\Pr(H \mid W) = 0.4$; and $\Pr(H \mid W^c) = 0.7$.

(a) $\Pr(W \cap H) = \Pr(H \mid W) \Pr(W) = 0.4 \times 0.3 = 0.12$.

(b) By the Bayes theorem,

$$\begin{aligned} \Pr(H) &= \Pr(H \mid W) \Pr(W) + \Pr(H \mid W^c) \Pr(W^c) \\ &= (0.4 \times 0.3) + (0.7 \times 0.7) \\ &= 0.61. \end{aligned}$$

(c) Let $H_j$ denote the event that she hits the target on the $j$th try. Then we want to know $\Pr(H_1 \cap H_2^c) + \Pr(H_1^c \cap H_2)$. If her attempts are independent from one another, then this is

$$\Pr(H_1) \Pr(H_2^c) + \Pr(H_1^c) \Pr(H_2) = (0.61 \times 0.39) + (0.39 \times 0.61) = 0.4758.$$

(d) Here, we want $\Pr(W^c \mid H^c)$. We can write this as

$$\frac{\Pr(W^c \cap H^c)}{\Pr(H^c)} = \frac{\Pr(W^c \cap H^c)}{0.39}.$$

Note that

$$\Pr(W^c \cap H^c) = \Pr(H^c \mid W^c) \Pr(W^c) = (1 - 0.7) \times 0.7 = 0.21.$$

Consequently,

$$\Pr(W^c \mid H^c) = \frac{0.21}{0.39} = \frac{21}{39} = \frac{7}{13}.$$

**2.** (a) Because $B \subseteq A$, the intersection of $A$ and $B$ is $B$. Therefore, $\Pr(A \mid B) = \Pr(B)/\Pr(B) = 1$.

(b) If $A \subseteq B$, then $A \cap B = A$ and $\Pr(A \mid B) = \Pr(A)/\Pr(B)$.

**3.** Let $H$ denote the event {heads}. We know that: $\Pr(H \mid C_1) = 1/3$; $\Pr(H \mid C_2) = 2/3$; and $\Pr(H \mid C_3) = 1$. Let us observe, using Bayes's theorem, that

$$\begin{aligned} \Pr(H) &= \Pr(H \mid C_1) \Pr(C_1) + \Pr(H \mid C_2) \Pr(C_2) + \Pr(H \mid C_3) \Pr(C_3) \\ &= \left(\frac{1}{3} \times \frac{1}{3}\right) + \left(\frac{2}{3} \times \frac{1}{3}\right) + \left(1 \times \frac{1}{3}\right) = \frac{2}{3}. \end{aligned}$$

- For the first portion, we are asked to compute $\Pr(C_k \mid H)$. But this is $\Pr(H \mid C_k)\Pr(C_k)/\Pr(H)$. Therefore,

$$\Pr(C_1 \mid H) = \frac{\frac{1}{3} \times \frac{1}{3}}{\frac{2}{3}} = \frac{1}{6},$$
$$\Pr(C_2 \mid H) = \frac{\frac{2}{3} \times \frac{1}{3}}{\frac{2}{3}} = \frac{1}{3},$$
$$\Pr(C_3 \mid H) = \frac{1 \times \frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}.$$

- Let $H_j$ denote the event that the $j$th attempt leads to heads. We are asked to compute $\Pr(H_2 \mid H_1)$. We may observe that

$$\begin{aligned}&\Pr(H_2 \mid H_1)\\ &= \Pr(H_2 \cap C_1 \mid H_1) + \Pr(H_2 \cap C_2 \mid H_1) + \Pr(H_2 \cap C_3 \mid H_1).\end{aligned}$$

Now:

$$\Pr(H_2 \cap C_1 \mid H_1) = \Pr(H_1 \cap H_2 \mid C_1)\,\frac{\Pr(C_1)}{\Pr(H_1)} = \left(\frac{1}{3}\right)^2 \frac{1/3}{2/3} = \frac{1}{18};$$
$$\Pr(H_2 \cap C_2 \mid H_1) = \Pr(H_1 \cap H_2 \mid C_2)\,\frac{\Pr(C_2)}{\Pr(H_1)} = \left(\frac{2}{3}\right)^2 \frac{1/3}{2/3} = \frac{2}{9};$$
$$\Pr(H_2 \cap C_3 \mid H_1) = \Pr(H_1 \cap H_2 \mid C_3)\,\frac{\Pr(C_3)}{\Pr(H_1)} = (1)^2 \frac{1/3}{2/3} = \frac{1}{2}.$$

Consequently,

$$\Pr(H_2 \mid H_1) = \frac{1}{18} + \frac{2}{9} + \frac{1}{2} = \frac{7}{9}.$$

- We first note that

$$\begin{aligned}\Pr(C_k \mid H_1 \cap H_2) &= \Pr(H_1 \cap H_2 \mid C_k)\ \frac{\Pr(C_k)}{\Pr(H_1 \cap H_2)}\\ &= \Pr(H_1 \cap H_2 \mid C_k)\ \frac{\Pr(C_k)}{\Pr(H_2 \mid H_1)\Pr(H_1)}\\ &= \Pr(H_1 \cap H_2 \mid C_k)\ \frac{\frac{1}{3}}{\frac{7}{9} \times \frac{2}{3}}\\ &= \frac{9}{14}\ \Pr(H_1 \cap H_2 \mid C_k).\end{aligned}$$

Therefore,

$$\Pr(C_1 \mid H_1 \cap H_2) = \frac{9}{14}\left(\frac{1}{3}\right)^2 = \frac{1}{14};$$
$$\Pr(C_2 \mid H_1 \cap H_2) = \frac{9}{14}\left(\frac{2}{3}\right)^2 = \frac{2}{7};$$
$$\Pr(C_3 \mid H_1 \cap H_2) = \frac{9}{14}(1)^2 = \frac{9}{14}.$$

- We proceed as before, and write

$$\begin{aligned}&\Pr(H_3 \mid H_1 \cap H_2)\\ &= \Pr(H_3 \cap C_1 \mid H_1 \cap H_2) + \Pr(H_3 \cap C_2 \mid H_1 \cap H_2)\\ &\quad + \Pr(H_3 \cap C_3 \mid H_1 \cap H_2).\end{aligned}$$

But

$$\begin{aligned}\Pr(H_3 \cap C_k \mid H_1 \cap H_2) &= \Pr(H_1 \cap H_2 \cap H_3 \mid C_k)\,\frac{\Pr(C_k)}{\Pr(H_1 \cap H_2)}\\ &= (\Pr(H \mid C_k))^3\ \frac{\frac{1}{3}}{\Pr(H_2 \mid H_1)\Pr(H_1)}\\ &= (\Pr(H \mid C_k))^3\ \frac{\frac{1}{3}}{\frac{7}{9}\times\frac{2}{3}}\\ &= \frac{9}{14}\ (\Pr(H \mid C_k))^3\,.\end{aligned}$$

Therefore,

$$\Pr(H_3 \cap C_1 \mid H_1 \cap H_2) = \frac{9}{14}\times\left(\frac{1}{3}\right)^3 = \frac{1}{42};$$
$$\Pr(H_3 \cap C_2 \mid H_1 \cap H_2) = \frac{9}{14}\times\left(\frac{2}{3}\right)^3 = \frac{9}{21};$$
$$\Pr(H_3 \cap C_3 \mid H_1 \cap H_2) = \frac{9}{14}\times(1)^3 = \frac{9}{14}.$$

Therefore,

$$\Pr(H_3 \mid H_1 \cap H_2) = \frac{1}{42} + \frac{9}{21} + \frac{9}{14} = \frac{6}{7}.$$

**4.** (a) $\Pr(A \cap A) = \Pr(A)$, but by independence, $\Pr(A \cap A) = |\Pr(A)|^2$. Therefore, $\Pr(A)$ is a solution to $x = x^2$. If $x \neq 0$, then we can divide by $x$ to find that $x = 1$. This proves that $\Pr(A)$ is zero or one.

(b) Because they are disjoint, $\Pr(A \cap B) = 0$. So unless $\Pr(A) = 0$ or $\Pr(B) = 0$, or both are zero, $A$ and $B$ cannot be independent. ["If $A$ happens, then we know that $B$ does not; and vice versa."]

(c) Because $B = (A \cap B) \cup (A^c \cap B)$ is a disjoint union,

$$\begin{aligned}\Pr(A^c \cap B) &= \Pr(B) - \Pr(A \cap B)\\ &= \Pr(B) - \Pr(A)\Pr(B)\\ &= \Pr(B)\,[1 - \Pr(A)]\\ &= \Pr(B)\Pr(A^c).\end{aligned}$$

Thus, $A^c$ and $B$ are independent. Similarly, $\Pr(A^c \cap B^c) + \Pr(A^c \cap B) = \Pr(A^c)$. Solve and apply the independence of $A^c$ and $B$ to find that

$$\Pr(A^c \cap B^c) = \Pr(A^c) - \Pr(A^c)\Pr(B) = \Pr(A^c)\Pr(B^c).$$

**14.** The probability in question is

$$\Pr\left(\bigcap_{j=1}^{n} A_j^c\right) = \prod_{j=1}^{n} \Pr(A_j^c) = \prod_{j=1}^{n} (1 - \Pr(A_j)),$$

where $\prod_{j=1}^{n} \alpha_j$ denotes the product $\alpha_1 \times \cdots \times \alpha_n$.

I claim that for

$$1 - x \le e^{-x} \quad \text{for all } x \ge 0. \tag{1}$$

If so, then

$$\Pr\left(\bigcap_{j=1}^{n} A_j^c\right) \le \prod_{j=1}^{n} e^{-\Pr(A_j)} = e^{-\sum_{j=1}^{n} \Pr(A_j)},$$

as desired. In order to prove (1) we apply Taylor's expansion with remainder to the function $f(x) = e^{-x}$, and find that

$$e^{-x} = 1 - x + \theta^2 \qquad \text{for some } 0 \le \theta \le x.$$

Because $\theta^2 \ge 0$, (1) follows.

**34.** Let $C = \{\text{cured}\}$. Then,

$$\Pr(C \mid A) = \frac{60}{200} = \frac{3}{10} \qquad \text{and} \qquad \Pr(C \mid B) = \frac{170}{1100} = \frac{17}{110}.$$

For the second part, let $M = \{\text{male}\}$ and $F = \{\text{female}\}$. We want to compute $\Pr(C \mid M \cap A)$, $\Pr(C \mid M \cap B)$, $\Pr(C \mid F \cap A)$, and $\Pr(C \mid F \cap B)$. We can see that $A$ is "better than" $B$ in the sense that $3/10 = 0.3$ is greater than $17/110 = 0.1\overline{54}$.

Then, we note that

$$\begin{aligned}
\Pr(C \mid M \cap A) &= \frac{50}{100} = \frac{1}{2}; \\
\Pr(C \mid M \cap B) &= \frac{60}{100} = \frac{3}{5}; \\
\Pr(C \mid F \cap A) &= \frac{10}{100} = \frac{1}{10}; \\
\Pr(C \mid F \cap B) &= \frac{110}{1000} = \frac{11}{100}.
\end{aligned}$$

Now $B$ looks "better than" $A$. For instance, for men, $1/2 = 0.5 < 3/5 = 0.6$, and for women, $1/10 = 0.1 < 11/100 = 0.11$.