Chapter 1 Problems

3.

(a) There are $15 \times 14 = 210$ different ways of selecting a pair. The number of ways that they can both be boys is $7 \times 6 = 42$. The number of ways that they are both girls is $8 \times 7 = 56$. Therefore, there are $42 + 56 = 98$ many ways in which the pair are of the same sex; $210 - 98 = 112$ many ways in which they are different. Thus,

$$\Pr\{\text{different sex}\} = \frac{112}{210} = \frac{8}{15}.$$

(b) There are $15 \times 15 = 225$ many ways of doing this altogether. The number of ways in which the children are both boys is $7 \times 7 = 49$; girls is $8 \times 8 = 64$. Thus, there are $49 + 64 = 113$ many ways in which the sexes are the same, and $225 - 113 = 112$ many ways in which they are different. Thus,

$$\Pr\{\text{different sexes}\} = \frac{112}{225}.$$

4. This was explained in lecture (Jan 19th).

7.

(a) $(A \cap B) \cup (B \cap C) \cup (A \cap B)$.
(b) $(A \cap B \cap C^c) \cup (B \cap C \cap A^c) \cup (A \cap B \cap C^c)$.
(c) $(A \cap B \cap C)^c$.
(d) $(A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C)$.

9.

(b) The probability of exactly three heads is

$$\begin{aligned}&\Pr(H_1H_2H_3T_4) + \Pr(H_1H_2T_3H_4) + \Pr(H_1T_2H_3H_4) \\ &\qquad + \Pr(T_1H_2H_3H_4) \\ &= \frac{1}{2^4} + \frac{1}{2^4} + \frac{1}{2^4} + \frac{1}{2^4} \\ &= \frac{4}{2^4} \\ &= \frac{1}{4}.\end{aligned}$$

(a) The probability of exactly four heads is $1/2^4 = 1/16$. Therefore, the probability of at least three heads is

$$\Pr\{\text{exactly three H}\} + \Pr\{\text{exactly four H}\} = \frac{1}{4} + \frac{1}{16} = \frac{5}{16}.$$

(c) We seek to find

$$\Pr(H_1H_2H_3H_4) + \Pr(H_1H_2H_3T_4) + \Pr(T_1H_2H_3H_4) = \frac{3}{16}.$$

12. The first portion is called the "inclusion-exclusion formula."

(a) When $n = 2$, this is merely the statement that

$$\Pr(A_1 \cup A_2) = \Pr(A_1) + \Pr(A_2) - \Pr(A_1 \cap A_2).$$

Let's try to derive the $n = 3$ case before we proceed in general.

$$\Pr(A_1 \cup \underbrace{A_2 \cup A_3}_{B}) = \Pr(A_1) + \Pr(B) - \Pr(A_1 \cap B).$$

By the $n = 2$ case,

$$\Pr(B) = \Pr(A_2) + \Pr(A_3) - \Pr(A_2 \cap A_3).$$

Also, note that

$$A_1 \cap B = (A_1 \cap A_2) \cup (A_1 \cap A_3).$$

Therefore,

$$\Pr(A_1 \cap B) = \Pr(A_1 \cap A_2) + \Pr(A_1 \cap A_3) - \Pr(A_1 \cap A_2 \cap A_3).$$

Putting these together, we obtain

$$\begin{aligned}\Pr(A_1 \cup A_2 \cup A_3) &= \Pr(A_1) + \Pr(A_2) + \Pr(A_3) \\ &\quad - \Pr(A_1 \cap A_2) - \Pr(A_1 \cap A_3) - \Pr(A_2 \cap A_3) \\ &\quad + \Pr(A_1 \cap A_2 \cap A_3).\end{aligned}$$

This is the inclusion-exclusion formula for $n = 3$.

The general case follows this very pattern. Can you fill in the gaps?

21. Note that

$$p = \frac{x(x-1)}{(x+y)(x+y-1)}.$$

(a) The equation $p = 1/2$ becomes

$$2x(x-1) = (x+y)(x+y-1).$$

That is,

$$2x^2 - 2x = x^2 + 2xy - x + y^2 - y.$$

Solve to obtain the quadratic equation,

$$x^2 - x(1+2y) - y(y-1) = 0. \tag{1}$$

If $y = 1$, then this becomes $x^2 - 3x = 0$. Since $x \geq 1$ is not zero, x has to be equal to three.

Now we study the case that y is even. If $y = 2$, then our equation (1) becomes

$$x^2 - 5x - 2 = 0.$$

The only solutions are

$$x = \frac{5 \pm \sqrt{25-8}}{2},$$

neither of which is an integer. The next highest possible even y is $y = 4$. In that case, (1) becomes

$$x^2 - 9x - 12 = 0.$$

The only solutions are

$$x = \frac{9 \pm \sqrt{81+48}}{2},$$

neither of which is an integer. The next possible even y is $y = 6$. In that case, (1) becomes

$$x^2 - 13x - 30 = 0.$$

The solutions are

$$x = \frac{13 \pm \sqrt{169+120}}{2} = \frac{13 \pm 17}{2}.$$

The positive solution is $x = (13+17)/2 = 15$, which is also an integer. Thus, when $y = 6$, $x = 15$.

(b) and (c) Similar to (a), and equally painful.