Solutions to Midterm #2
Mathematics 5010–1, Spring 2006
Department of Mathematics, University of Utah

1. *An urn contains R red balls and W white balls, where R and W are strictly positive integers. Balls are drawn at random one after another. Every time a ball is drawn, it is replaced back in the urn together with N new balls of the same color.*

(a) *Compute the probability that a red ball is drawn on the first draw.*

Solution: $R/(R+W)$.

(b) *Compute the probability that a red ball is drawn on the second draw.*

Solution: Define $R_i := \{\text{red on the } i\text{th draw}\}$, and $W_i := \{\text{white on the } i\text{th draw}\}$. Then, according to Bayes' formula,

$$\begin{aligned} P(R_2) &= P(R_2|R_1)P(R_1) + P(R_2|W_1)P(W_1) \\ &= \frac{R+N}{R+W+N} \cdot \frac{R}{R+W} + \frac{R}{R+W+N} \cdot \frac{W}{R+W} \\ &= \frac{R}{(R+W+N)(R+W)} \cdot (R+N+W) \\ &= \frac{R}{R+W}. \end{aligned}$$

2. *In a certain town, 60 percent of all property owners oppose an increase in the property tax while 80 percent of non-property owners favor it. If 65 percent of all registered voters are property owners, then what proportion of registered voters favor the tax increase?*

Solution: Let $O := \{\text{property owner}\}$ and $F := \{\text{favor tax increase}\}$. Then, $P(O) = 0.65$, $P(F^c|O) = 0.6$, and $P(F|O^c) = 0.8$. Thus,

$$P(F) = \underbrace{P(F|O)}_{0.4}\,\underbrace{P(O)}_{0.65} + \underbrace{P(F|O^c)}_{0.8}\,\underbrace{P(O^c)}_{0.35} = 0.54.$$

3. *A pair of fair dice are cast, and the number of rolled dots, on each die, is recorded. Let X denote the maximum of the two numbers. Find the probability mass function of X.*

Solution: Let $P(i\,,j)$ be the probability that first we roll an i, and then a j. Of course,

$P(i\,,j) = 1/36$ for all $i, j = 1, \ldots, 6$. Then,

$$P\{M = 1\} = P(1\,,1) = \frac{1}{36}$$
$$P\{M = 2\} = P(1\,,2) + P(2\,,1) + P(2\,,2) = \frac{3}{36}$$
$$P\{M = 3\} = P(1\,,3) + P(3\,,1) + P(2\,,3) + P(3\,,2) + P(3\,,3) = \frac{5}{36}$$
$$P\{M = 4\} = P(1\,,4) + \cdots + P(4\,,4) = \frac{7}{36}$$
$$P\{M = 5\} = P(1\,,5) + \cdots + P(5\,,5) = \frac{9}{36}$$
$$P\{M = 6\} = P(1\,,6) + \cdots + P(6\,,6) = \frac{11}{36}.$$

4. *A fair coin is cast until the first head appears. Let N denote the number of tosses needed to see the first head.*

(a) *Find the probability mass function of N.*

Solution: For all $k = 1, 2, \ldots,$

$$P\{N = k\} = \frac{1}{2^k}.$$

For all other values of k, $P\{N = k\} = 0$.

(b) *Compute $P\{3 \leq N < 8\}$.*

Solution: We want

$$P\{3 \leq N < 8\} = P\{M = 3\} + \cdots + P\{M = 7\}$$
$$= \frac{1}{2^3} + \cdots + \frac{1}{2^7}.$$

5. *Here is the simplest mathematical model for the evolution of the price of a commodity: At time zero, the value is zero. Then at every time-step ($n = 1, 2, \cdots$), the stock-price goes up or down by one unit with probability $1/2$. If all stock movements are independent of one another, then compute the probability that the value is zero at time $2n$.*

Solution: Let U denote the number of times that the stock value goes up in $2n$ time-steps, and D the number of corresponding downs. Note that: (i) $D = 2n - U$; and (ii) we are at zero at time n if and only if $D = U$. Therefore, we seek the probability $P\{2n - U = U\} = P\{U = n\}$. This is a binomial probability: $P\{U = n\} = \binom{2n}{n}(\frac{1}{2})^{2n}$.