1. An urn contains $R$ red balls and $W$ white balls, where $R$ and $W$ are strictly positive integers. Balls are drawn at random one after another. Every time a ball is drawn, it is replaced back in the urn together with $N$ new balls of the same color.

(a) Compute the probability that a red ball is drawn on the first draw.

Solution: \[ \frac{R}{R+W}. \]

(b) Compute the probability that a red ball is drawn on the second draw.

Solution: Define $R_i := \{\text{red on the } i\text{th draw}\}$, and $W_i := \{\text{white on the } i\text{th draw}\}$. Then, according to Bayes’ formula,

\[
P(R_2) = P(R_2|R_1)P(R_1) + P(R_2|W_1)P(W_1)
\]

\[
= \frac{R+N}{R+W+N} \cdot \frac{R}{R+W} + \frac{R}{R+W+N} \cdot \frac{W}{R+W}
\]

\[
= \frac{R}{(R+W+N)(R+W)} \cdot (R+N+W)
\]

\[
= \frac{R}{R+W}.
\]

2. In a certain town, 60 percent of all property owners oppose an increase in the property tax while 80 percent of non-property owners favor it. If 65 percent of all registered voters are property owners, then what proportion of registered voters favor the tax increase?

Solution: Let $O := \{\text{property owner}\}$ and $F := \{\text{favor tax increase}\}$. Then, $P(O) = 0.65$, $P(F^c|O) = 0.6$, and $P(F|O^c) = 0.8$. Thus,

\[
P(F) = P(F|O)P(O) + P(F|O^c)P(O^c) = 0.54.
\]

3. A pair of fair dice are cast, and the number of rolled dots, on each die, is recorded. Let $X$ denote the maximum of the two numbers. Find the probability mass function of $X$.

Solution: Let $P(i,j)$ be the probability that first we roll an $i$, and then a $j$. Of course,
\[ P(i, j) = 1/36 \text{ for all } i, j = 1, \ldots, 6. \] Then,

\[
\begin{align*}
P(M = 1) &= P(1, 1) = \frac{1}{36} \\
P(M = 2) &= P(1, 2) + P(2, 1) + P(2, 2) = \frac{3}{36} \\
P(M = 3) &= P(1, 3) + P(3, 1) + P(2, 3) + P(3, 2) + P(3, 3) = \frac{5}{36} \\
P(M = 4) &= P(1, 4) + \cdots + P(4, 4) = \frac{7}{36} \\
P(M = 5) &= P(1, 5) + \cdots + P(5, 5) = \frac{9}{36} \\
P(M = 6) &= P(1, 6) + \cdots + P(6, 6) = \frac{11}{36}.
\]

4. A fair coin is cast until the first head appears. Let \( N \) denote the number of tosses needed to see the first head.

(a) Find the probability mass function of \( N \).

Solution: For all \( k = 1, 2, \ldots \),

\[ P\{N = k\} = \frac{1}{2^k}. \]

For all other values of \( k \), \( P\{N = k\} = 0 \).

(b) Compute \( P\{3 \leq N < 8\} \).

Solution: We want

\[
P\{3 \leq N < 8\} = P\{M = 3\} + \cdots + P\{M = 7\} = \frac{1}{2^3} + \cdots + \frac{1}{2^7}.
\]

5. Here is the simplest mathematical model for the evolution of the price of a commodity: At time zero, the value is zero. Then at every time-step \( (n = 1, 2, \ldots) \), the stock-price goes up or down by one unit with probability \( 1/2 \). If all stock movements are independent of one another, then compute the probability that the value is zero at time \( 2n \).

Solution: Let \( U \) denote the number of times that the stock value goes up in \( 2n \) time-steps, and \( D \) the number of corresponding downs. Note that: (i) \( D = 2n - U \); and (ii) we are at zero at time \( n \) if and only if \( D = U \). Therefore, we seek the probability \( P\{2n - U = U\} = P\{U = n\} \). This is a binomial probability:

\[ P\{U = n\} = \binom{2n}{n} \left(\frac{1}{2}\right)^{2n}. \]