Solutions to Midterm #1
Mathematics 5010–1, Spring 2006
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1. *When you walk into a certain sandwich shop, you will be asked, "Would like it on wheat or white bread"? After you answer, then you are asked, "Would you like it with ham, beef, or salami"? After you answer, then you are asked, "Which of the following condiments would you like on your sandwich: pickles, tomato, onion"? Finally, you are asked "would you like cheese on that"? At this point, you will be given a sandwich made to your specifications. How many different sandwich combinations does this shop offer?*

Solution: You have to perform: Choose a bread (2 ways); choose a meat (3 ways); choose a condiment (8 ways—this is the number of subsets of three items); decide on cheese (2 ways). Total $= 2 \times 3 \times 8 \times 2 = 96$ possible sandwiches. If you want to have "no meat" as an option, then the answer is $2 \times 4 \times 8 \times 2 = 128$ sandwiches instead.

2. *Suppose* 8 *new teachers are to be divided among* 4 *schools.*

a. *How many divisions are possible?*

Solution: $4^8 = 65,536$.

b. *Repeat the preceding, but now assume that each school must receive* 2 *teachers?*

Solution: Either $\frac{8!}{2!2!2!2!} = 2,520$, or $\binom{8}{2}\binom{6}{2}\binom{4}{2} = 2,520$.

3. *Prove that the following combinatorial identities hold for all integers $n \geq 0$,*

a. $\sum_{j=0}^{n} \binom{n}{j} = 2^n$;

b. $\sum_{j=0}^{n} (-1)^j \binom{n}{j} = 0$;

c. $\sum_{j=0}^{n} (-2)^j \binom{n}{j} = (-1)^n$.

Solution: Recall the *binomial theorem*: For all real numbers a, b, and all integers $n \geq 1$,

$$(a+b)^n = \sum_{j=0}^{n} \binom{n}{j} a^j b^{n-j}.$$

To prove **a** set $a = b = 1$. To prove **b** set $a = -1$ and $b = 1$. To prove **c** set $a = -2$ and $b = 1$.

4. *There are n people in a room. If all pairs shake hands, then how many handshakes will take place?*

Solution: $\binom{n}{2} = \frac{n(n-1)}{2}$.

5. *How many jumbles of "Mississippi" are there?*

Solution: $\frac{11!}{4!\cdot 4!\cdot 2!} = 34,650$.