Gambler's ruin

$$P_{j+1} - P_{-M+1} = \frac{q}{p} P_{-M+1} \sum_{k=0}^{M+j-1} \left(\frac{q}{p}\right)^k$$

$$\sum_0^N r^k = \frac{1-r^{N+1}}{1-r}. \quad \text{So}$$

$$P_{j+1} - P_{-M+1} = \frac{q}{p} P_{-M+1} \cdot \frac{1-(q/p)^{M+j}}{1-(q/p)}.$$

$$P_{j+1} = P_{-M+1}\left[1 + \frac{q}{p}\frac{1-(q/p)^{M+j}}{1-(q/p)}\right]$$

$$= P_{-M+1} \cdot \left[1 + \frac{q - q(q/p)^{M+j}}{p-q}\right]$$

$$= P_{-M+1} \cdot \frac{p - q(q/p)^{M+j}}{p-q}. \qquad \oplus$$

$$1 = P_N = P_{-M+1} \cdot \frac{p-q(q/p)^{M+N-1}}{p-q}$$

$$\Rightarrow \quad P_{-M+1} = \frac{p-q}{p-q(q/p)^{M+N-1}}$$

By $\oplus$

$$P_0 = \frac{p-q}{p-q(q/p)^{M+N-1}} \times \frac{p-q(q/p)^{M-1}}{p-q}$$

$$= \frac{p-q(q/p)^{M-1}}{p-q(q/p)^{M+N-1}}.$$

$$= \frac{p^M - q^M}{p^M - q^{M+N}/p^N} = \frac{p^M - q^M}{p^M - q^M\left(\frac{q}{p}\right)^N}.$$

If $p > \frac{1}{2}$, then $q/p < 1$ and $(q/p)^N \to 0$ as $N \to \infty$.

So
$$P_0 \to \frac{p^M - q^M}{p^M} = 1 - \left(\frac{q}{p}\right)^M$$

= prob. of winning ∞ before losing M.

Eg. $P\{\text{ever losing } 1\} = 1 - \left(1 - \frac{q}{p}\right) = q/p$.

If $p < \frac{1}{2}$, by symmetry, then $P(\text{ever winning}) = p/q$.

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Ex. [<u>Laplace Rule of Succession</u>]

$k+1$ coins in a box. The ith flips H with prob i/k. $0 \le i \le k$.

Choose a coin at random and flip it n times independently.

If the 1st are all H's, what is the cond'l prob. that the $(n+1)$st is H?

$C_i = \{\text{coin } i\}$ $\qquad$ $F_i = \{\text{first } i \text{ are H}\}$.

$\cancel{P(H|F_n)}$ $\qquad$ $\cancel{P(F_{n+1}|F_n) = \sum_{j=0}^{n} P(F_{n+1}|F_j C) P(F_j C)}$

$$P(F_{n+1}|F_n) = \sum_{j=0}^{k} P(F_{n+1}|F_n C_j) P(C_j|F_n) \qquad \text{[why?]}$$

$$= \sum_{j=0}^{k} P(H|C_j) \, P(C_j|F_n)$$

$$= \sum_{j=0}^{k} (j/k) \, P(C_j|F_n)$$

$$P(C_j|F_n) = P(F_n|C_j)\frac{P(C_j)}{P(F_n)} = \frac{(j/k)^n \frac{1}{k+1}}{P(F_n)}$$

$$P(F_n) = \sum_{l=0}^{k} P(F_n \mid C_l)\, P(C_l) = \frac{1}{k+1} \sum_{l=0}^{k} \left(l/k\right)^n .$$

$$\therefore \quad P(C_j \mid F_n) = \frac{(j/k)^n \frac{1}{k+1}}{\frac{1}{k+1} \sum_{l=0}^{k} (l/k)^n} = \frac{(j/k)^n}{\sum_{l=0}^{k} (l/k)^n}$$

$$\therefore \quad P(F_{n+1} \mid F_n) = \frac{\sum_{j=0}^{k} (j/k)^{n+1}}{\sum_{j=0}^{k} (j/k)^n} .$$

$$\xrightarrow{k \to \infty} \frac{\int_0^1 x^{n+1}\, dx}{\int_0^1 x^n\, dx} = \frac{n+1}{n+2} .$$

Example (The problem of the points)

Bernoulli trials $p = P(\text{Success})$.

$P(n \text{ successes before } m \text{ failures}) = ?$

Look at the first $n+m-1$ trials. n success before m failure iff $\exists$ n or more successes in the 1^{st} $n+m-1$ trials.

$\therefore$ prob. n successes before m failures

$$= p(n) + p(n+1) + \dots + p(n+m-1)$$

$$= \binom{n+m-1}{n} p^n (1-p)^{m-1} + \dots + \binom{n+m-1}{n+m-1} p^{n+m-1} (1-p)^0$$

$$= \sum_{j=n}^{n+m-1} \binom{n+m-1}{j} p^j (1-p)^{n+m-1-j} .$$