① Recall indep't events : 1st for 2 events; next, all subsets of $\{E_1, \dots, E_n\}$ of size $n-1$ or less should be indep't.

---

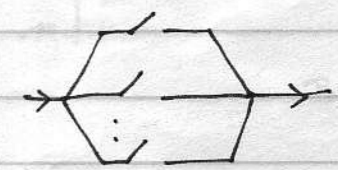
② Ex. E_i indep't $P(E_i) = p \;\; \forall i \Rightarrow$

$$P(\text{all } E_i\text{'s occur}) = \lim_{n\to\infty} P(E_1 \cap \dots \cap E_n) \quad (\text{conty of prob's})$$

$$= \lim_{n\to\infty} p^n = 0, \quad \text{unless } p=1 \,!$$

---

③. Ex. n parts in a circuitry ; all die with prob p.
Parallel or series ?



$P[\text{doesn't work}] = p^n$ $\qquad$ $P[\text{doesn't work}] = 1-(1-p)^n$.

When is $1-(1-p)^n > p^n$?

i.e., when is $1-p^n \geq (1-p)^n$? Always:

$n=2$: $(1-p)^2 = 1-2p+p^2$

$(1-p)^2 - (1-p^2) = 2p^2 - 2p = 2p(p-1) \leq 0$

Suppose $(1-p)^n \leq 1-p^n$; prove it for $n+1$:

$$(1-p)^{n+1} = (1-p)^n(1-p) \leq (1-p^n)(1-p)$$

$$= 1-p-p^n+p^{n+1}.$$

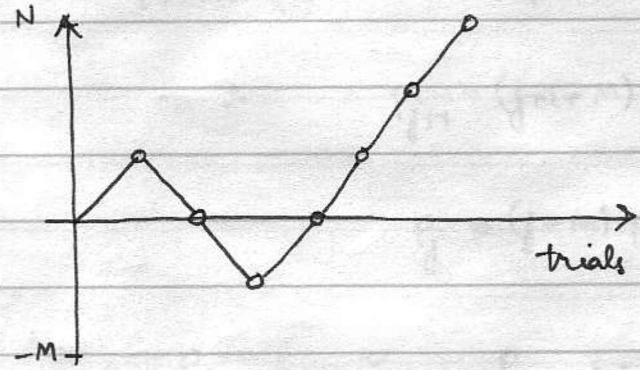
goal : $p + p^n - p^{n+1} \geq p^{n+1}$

$\Longleftarrow \quad p + p^n \geq 2p^{n+1}$. But $p + p^n \geq p^{n+1} + p^{n+1}$!

---

④ Ex. (Gambler's Ruin) Fair games played independently.

What is the prob of winning \$N before losing \$1 ?



$P_i = \text{prob}\{\underbrace{\text{Hit } N \text{ before } -M}_{H_{NM}} \mid \text{at time zero fortune} = i\}$.

Question : Find P_0.

But $P_0 = P[H_{NM} \mid W_1 = 1]P[W_1 = 1] + P[H_{NM} \mid W_1 = -1]P[W_1 = -1]$

$$= \frac{1}{2} P_1 + \frac{1}{2} P_{-1}.$$

More generally, if $-M < i \leq N$, then

$$P_i = \frac{1}{2} P_{i+1} + \frac{1}{2} P_{i-1}. \qquad ⊛$$

Also $P_N = 1$, $P_{-M} = 0$. "Boundary values."

~~By ⊛, if $-M < i \leq N$, then~~

$$\cancel{P_{i+1} - P_i = \frac{1}{2}(P_{i+1} - P_{i-1})} \qquad ⊛⊛$$

$$\cancel{\sum_{i=-M+1}^{N-1} (P_{i+1} - P_i) = \frac{1}{2} \sum_{i=-M+1}^{N-1} (P_{i+1} - P_{i-1})}.$$

Use $p_i = \frac{p_i}{2} + \frac{p_i}{2}$ to get

$$p_{i+1} - p_i = p_i - p_{i-1} \qquad -M < i < N. \quad (**)$$

So, $p_{i+1} - p_i = p_{i-1} - p_{i-2} = \cdots = p_{-M+1} - p_{-M} = p_{-M+1}$.

$$\sum_{i=-M+1}^{j} (p_{j+1} - p_i) = p_{j+1} - p_{-M+1} = (j+M)\, p_{-M+1}.$$

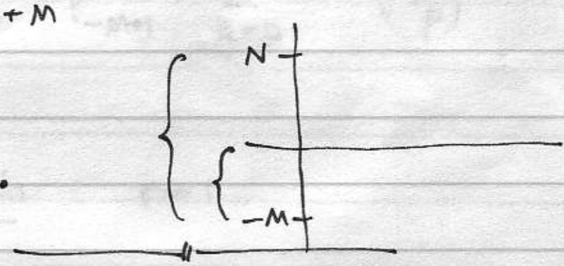
$$\therefore \quad p_{j+1} = (j+1+M)\, p_{-M+1} \qquad -M+1 \le j \le N-1$$

$$\longleftrightarrow \quad p_j = (j+M)\, p_{-M+1}, \qquad -M+2 \le j \le N.$$

Also, $p_N = 1$ so $p_{M+1} = \frac{1}{N+M}$

so $p_j = \frac{j+M}{N+M}$.

Ans. $p_0 = \frac{M}{N+M}$.



(5) Ex (Gambler's ruin) If $P(\text{win}) = p$ $P(\text{lose}) = 1-p$ per game and $p \neq \frac{1}{2}$, then what? Let $q = 1-p$. Then

$$p_i = p\, p_{i+1} + q\, p_{i-1}$$

$p_i = p\, p_i + q\, p_i$, so

$$p(p_{i+1} - p_i) = q(p_i - p_{i-1}), \qquad -M+1 \le i \le N-1.$$

$$\therefore \quad P_{i+1}-P_i = \left(\frac{q}{p}\right)(P_i - P_{i-1})$$

$$= \left(\frac{q}{p}\right)^2 (P_{i-1}-P_{i-2}) = \left(\frac{q}{p}\right)^3 (P_{i-2}-P_{i-3})$$

$$= \cdots = \left(\frac{q}{p}\right)^{k+1} (P_{i-k}-P_{i-k-1}) .$$

Want $i-k-1=-M$. Thus, $k = M+i-1$:

$$P_{i+1}-P_i = \left(\frac{q}{p}\right)^{M+i} P_{-M+1} \qquad (P_{-M}=0).$$

$$P_{j+1}-P_{-M+1} = \sum_{i=-M+1}^{j} (P_{i+1}-P_i) = P_{-M+1} \cdot \sum_{i=-M+1}^{j} \left(\frac{q}{p}\right)^{M+i}$$

$$= P_{-M+1} \cdot \sum_{k=0}^{M+j-1} \left(\frac{q}{p}\right)^{k+1} \qquad (k = i+M-1)$$

$$= \frac{q}{p} \cdot P_{-M+1} \cdot \sum_{k=0}^{M+j-1} \left(\frac{q}{p}\right)^{k}.$$

<u>Aside on geometric series</u> $\quad r \neq 1$

$$S_N = \sum_{i=0}^{N} r^i$$

$$- \quad rS_N = S_{N+1} - 1$$

$$- \quad S_{N+1} = S_N + r^{N+1}$$

$$\therefore \quad rS_N = S_N + r^{N+1} - 1$$

$$\therefore \quad S_N = \frac{1-r^{N+1}}{1-r} .$$