Examples of Conditional Probability

Ex. 1

Let \( S = \{ \text{survives the delivery} \} \) and \( C = \{ \text{had a C-section} \} \).

Given the sample space, \( P(S) = 0.98 \), \( P(C) = 0.15 \), and \( P(S | C) = 0.96 \).

Q: Find \( P(S | C) \).

\[
P(S | C) = \frac{P(S \cap C)}{P(C)}
\]

\[
= P(S) - P(S | C) \cdot P(C)
\]

\[
= 0.98 - 0.96 \cdot 0.15
\]

\[
= 0.836
\]

Ex. 2

Let \( F = \{ \text{female} \} \) and \( CS = \{ \text{computer science major} \} \).

\( P(F) = 0.52 \), \( P(CS) = 0.05 \), and \( P(F \cap CS) = 0.02 \).

Q1: \( P(F | CS) = \frac{0.02}{0.05} = 0.4 \)

Q2: \( P(CS | F) = \frac{P(F \cap CS)}{P(F)} = \frac{0.02}{0.52} \approx 0.0385 \)

\[
\text{Note:}\quad P(CS | F) = \frac{P(CS)}{P(F)} = \frac{0.05}{0.52}
\]
Ex. 3.

*Ex 33, p. 107* (a)

A fair coin, \( \square \), and a 2-headed coin.

Pick a coin at random and flip it. \( P(\text{fair} \mid H) = ? \)

\[
P(\text{fair} \mid H) = \frac{P(\text{fair} \cap H)}{P(H)}
\]

\[
P(\text{fair} \cap H) = P(H \mid \text{fair}) \cdot P(\text{fair}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.
\]

\[
P(H) = P(H \mid \text{fair}) \cdot P(\text{fair}) + P(H \mid \text{unfair}) \cdot P(\text{unfair})
\]

\[
= \frac{1}{2} \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{3}{4}.
\]

\[
\Rightarrow P(\text{fair} \mid H) = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}.
\]

(Aside)

We've found the following:

**Lemma.**

\[
P(A \mid B) = \frac{P(A \cap B)}{P(B)}.
\]

**Proof.**

\[
P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(\text{B} \mid A) \cdot P(A)}{P(B)} = \frac{P(B \mid A) \cdot P(A)}{P(B)}. \quad \#
\]

Ex. 4.

*Ex 33, p. 107* (b)

Suppose the gambler flips the same coin, flips it again, and again gets 'H'. \( P(\text{fair} \mid H_H_2) = ? \)

\[
P(H_H_2 \mid \text{fair}) = \frac{1}{4}.
\]

\[
P(\text{fair}) = \frac{1}{2} \quad \text{so need } P(H_1 \cap H_2).
\]

\[
P(H_H_2) = P(H_H_2 \mid \text{fair}) \cdot P(\text{fair}) + P(H_H_2 \mid \text{unfair}) \cdot P(\text{unfair})
\]

\[
= \frac{1}{4} \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{5}{8} \quad \therefore P(\text{fair} \mid H_H_2) = \frac{\frac{1}{4}}{\frac{5}{8}} = \frac{1}{5}.
\]
Ex. 5

**Problem:** Suppose the gambler flips the same coin but \( n \) times, and gets all heads. Find the probability of this event:

\[
P(H_n \cap \ldots \cap H_n | \text{fair}) = \frac{1}{2^n}.
\]

**Solution:**

\[
P(H_n \cap \ldots \cap H_n | \text{fair}) = \frac{1}{2^n}.
\]

\[
P(\text{fair}) = \frac{1}{2}.
\]

\[
P(H_n \cap \ldots \cap H_n) = P(H_n | \text{fair}) P(\text{fair}) + P(H_n | \text{unfair}) P(\text{unfair})
\]

\[
= \frac{1}{2^n} \cdot \frac{1}{2} + \frac{1}{2}
\]

\[
= \frac{1}{2} \left( \frac{1}{2^n + 1} \right) = \frac{2^n + 1}{2^{n+1}} = \frac{2^{n+1}}{2^{n+1}}.
\]

**Conclusion:**

\[
P(\text{fair} | H_n \cap \ldots \cap H_n) = \frac{1}{2^n} \cdot \frac{1}{2^{n+1}} = \frac{1}{2^n} \cdot \frac{1}{2^{n+1}} = \frac{1}{2^n} \cdot \frac{1}{2^{n+1}}.
\]

\[
\text{as } n \to \infty.
\]

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**Ex. 6**

An insurance company has 3 classifications: good (g), average (a), and bad (b) risks.

\[
P(\text{accident} | g) = 0.05
\]

\[
P(\text{accident} | a) = 0.15
\]

\[
P(\text{accident} | b) = 0.3.
\]

\[
P(g) = 0.2, \quad P(a) = 0.5, \quad P(b) = 0.3.
\]

**Problem:** Find \( P(g | \text{no accident}) \). Let \( A := \{ \text{accident} \} \).
\[ P(g|A^c) = P(A^c|g) \frac{P(g)}{P(A^c)} \]

(i) \[ P(A^c|g) = 1 - P(A|g) = 0.95 \quad \left[ P(A^c|a) = 0.85 ; P(A^c|b) = 0.7 \right] \]

(ii) \[ P(g) = 0.2 \quad \left[ P(a) = 0.5 ; P(b) = 0.3 \right] \]

(iii) \[ P(A) = P(A|g)P(g) + P(A|a)P(a) + P(A|b)P(b) = 0.265 \]

\[ \therefore P(A^c) = 1 - 0.265 = 0.735 \]

\[ \Rightarrow P(g|A^c) = 0.2585 \quad \text{What is going on?} \]

\[ P(a|A^c) = 0.578 \quad \left[ P(b|A^c) = 0.2857 \right] \]