Conditional Probabilities & Independence

Ex. 30 females — 10 of the women work
50 males — 15 — " — men work

$\#\{\text{male}\}/_{80} = \mathbb{P}(\text{male})$ — if 1 person is picked at random.

$= 5/8$

$\mathbb{P}(\text{female}) = 3/8$

$\mathbb{P}(\text{work}) = 25/80$

$\mathbb{P}(\text{not work}) = 55/80$

$\mathbb{P}(\text{Male \& works}) = \frac{15}{80}$

$\mathbb{P}(\text{female \& works}) = \frac{10}{80}$.

$$\mathbb{P}\{\text{male} \mid \text{works}\} = \frac{\#15}{25} = \frac{15/80}{25/80} = \frac{\mathbb{P}(\text{male \& works})}{\mathbb{P}(\text{works})}.$$

given that you know the person works

In general, $$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

Ex. A couple has 2 kids. If all sexes are equally likely, find

$\mathbb{P}(g_1 \cap g_2 \mid g_1)$ — $g_i = i^{\text{th}}$ kid is a girl

$$\mathbb{P}(g_1 \cap g_2 \mid g_1) = \frac{\mathbb{P}(g_1, g_2 \cap g_1)}{\mathbb{P}(g_1)} = \frac{\mathbb{P}(g_1, g_2)}{\mathbb{P}(g_1)} = \frac{\#\{g_1, g_2\}/4}{\#\{g_1\}/4}$$

$$= \frac{1/4}{2/4} = \frac{1}{2}.$$

Ex. Two classes : one test.

class 1 (30 students)		class 2 (30 students)	
15	good score	5	good
10	average	10	average
5	poor	15	poor

pick a student @ random from each. Find

average

$$\mathbb{P}\Big\{\underbrace{\text{student 1 came from class 1}}_{A} \;\Big|\; \underbrace{\text{student 1 = fair , student 2 = poor}}_{B}\Big\}$$

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)}$$

~~P(A|B) =~~

$$= \frac{10\times 15 \,/\, 30\times 30}{\dfrac{10\times 15 + 5\times 10}{30\times 30}}$$

$$= \frac{150}{150+50} = \frac{15}{20} = \frac{3}{4}\,.$$

Ex. (Cards). Deal 2 cards @ random. Find $\mathbb{P}(K_1\cap K_2)$.

$$\mathbb{P}(K_1\cap K_2) = \mathbb{P}(K_2|K_1)\,\mathbb{P}(K_1)$$

$$= \frac{3}{51}\times\frac{4}{52}$$

$$\mathbb{P}(K_1\cap K_2\cap K_3) = \mathbb{P}(K_3|K_1\cap K_2)\,\mathbb{P}(K_1\cap K_2)$$

$$= \frac{2}{50}\times\frac{3}{51}\times\frac{4}{52}\,.$$

Theorem: Conditional prob's are prob's in their first argument.

"Proof" $- \; P(\emptyset | E) = \dfrac{P(\emptyset \cap E)}{P(E)} = 0 \quad ; \quad P(S|E) = \dfrac{P(S \cap E)}{P(E)} = 1$

$$- \; P(A^c | E) = \frac{P(A^c \cap E)}{P(E)} = \frac{P(E) - P(A \cap E)}{P(E)}$$

$$= 1 - P(A|E)$$

$$- \; P(A \cup B | E) = \frac{P(A \cap E \cup B \cap E)}{P(E)}$$

$$= P_E(A) + P_E(B) - P_E(A \cap B).$$

$-$ "σ-additive" #

Theorem. $E_1, E_2, \ldots, E_N$ disjoint ; $\bigcup_{j=1}^{N} E_j = S$;

$$\Rightarrow \quad P(A) = \sum_{j=1}^{N} P(A|E_j)\, P(E_j).$$

Proof. $A = (A \cap E_1) \cup \ldots \cup (A \cap E_N)$; a disjoint union

$$\therefore \; P(A) = \sum_{j=1}^{N} P(A \cap E_j) = \sum_{j=1}^{N} P(A|E_j)\, P(E_j). \qquad \#$$

Ex. [The Monte Hall Problem]

Three closed doors; big prize behind one of them.
You choose a door at random. Before opening it, the host opens another one (w/o the prize) and gives you the option of: (i) Either opening your door; or (ii) switching.
Should you switch?

$W = \{\text{win}\}$ $\quad$ $C = \{\text{chose the correct door at the beginning}\}$.

If no switch, then $P(W) = P(C) = 1/3$.
If switch, then $P(W) = P(W|C)\,P(C) + P(W|C^c)\,P(C^c)$

$$= 0 + 1 \times 2/3 = 2/3.$$

Switch.

Ex. Deal 2 cards: $P(K_2) = P(K_2 | K_1^c)\,P(K_1^c) + P(K_2 | K_1)\,P(K_1)$

$$= \frac{4}{51} \cdot \frac{48}{52} + \frac{3}{51} \cdot \frac{4}{52}$$

$$= \frac{4}{52 \times 51}\left[48 + 3\right] = \frac{4}{52} = P(K_1).$$

Likewise, $P(A_2) = P(A_1)$ etc. $= 4/52$.