Ex. (B-Day Problem) n people in a room. All B-Day possibilities equally likely, find $\mathbb{P}$(no 2 of them have the same B-Day)

$S =$ (all B-Day choices) $\qquad \#(S) = 365^n$

~~$\mathbb{P}$(no 2 have the same) = 1 − $\mathbb{P}$(~~

$$\#\{\text{no 2 have the same}\} = \binom{365}{n} \quad n!$$

choose the different days — give them out

$$\mathbb{P}(-\,''\,-) = \frac{\binom{365}{n}\, n!}{365^n} = \alpha(n).$$

N.B.: $\alpha(20) =$ ~~[illegible]~~ 0.5563

$\alpha(25) = 0.4018$

$\alpha(30) = 0.2695$

$\alpha(50) = 0.0256$

$\alpha(70) = 0.00022 \qquad \alpha(100) = 0.00000023$

Ex. Want to put n balls in N urns. Find $\mathbb{P}$ {exactly m balls in the 1st urn}.

Assume all possible arrangements are equally likely.

\# of arrangements = ball #1 ○ (N), ball #2 ○ (N), …, ball #n ○ (N) $\qquad \# = N^n$.

$$\#(\text{of ways to put exactly } m \text{ balls in urn } \#1) = \binom{n}{m} \quad (N-1)^{n-m}$$

choose the m — put the other $n-m$ balls in the other $(N-1)$ urns.

$$\mathbb{P}(\text{exactly } m \text{ in urn } \#1) = \frac{\binom{n}{m}(N-1)^{n-m}}{N^n}.$$

Poincaré: Inclusion-Exclusion Identity

$E_1, \dots, E_N$: events. $(N \geq 2)$

$$P\left(\bigcup_{j=1}^{N} E_j\right) = \sum_{j=1}^{N} P(E_j) - \sum\sum_{i<j} P(E_i \cap E_j)$$

$$+ \sum\sum\sum_{i<j<k} P(E_i \cap E_j \cap E_k) + \cdots$$

$$+ (-1)^{\ell+1} \sum_{i_1 < \cdots < i_\ell} \cdots \sum P(E_{i_1} \cap \cdots \cap E_{i_\ell}) + \cdots$$

$$+ (-1)^{N+1} P(E_1 \cap \cdots \cap E_N).$$

<u>Proof</u> <u>$N=2$</u>:

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2) \quad . \quad \checkmark$$

<u>$N=3$</u>: $P(E_1 \cup E_2 \cup E_3) = P(E_1 \cup E_2) + P(E_3) - P((E_1 \cup E_2) \cap E_3)$

$$= P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2) - P(E_1 \cap E_3 \cup E_2 \cap E_3)$$

$$= \sum_{i=1}^{3} P(E_i) - P(E_1 \cap E_2) - P(E_1 \cap E_3) - P(E_2 \cap E_3) + P(E_1 \cap E_2 \cap E_3)$$

$$= \sum_{i=1}^{3} P(E_i) - \sum\sum_{i<j} P(E_i \cap E_j) + P(E_1 \cap E_2 \cap E_3). \quad \checkmark$$

Suppose the formula holds for N. Show it holds for $N+1$:

$$P\left(\bigcup_{i=1}^{N+1} E_i\right) = P\left(\bigcup_{i=1}^{N} E_i\right) + P\left(\cancel{\bigcup_{i=1}^{N} E_i \cap}\, E_{N+1}\right) - P\left(\bigcup_{i=1}^{N} E_i \cap E_{N+1}\right)$$

$$= \sum_{i=1}^{N+1} P(E_i) + \cdots + (-1)^{\ell+1} \sum_{i_1<\cdots<i_\ell \le N} \cdots \sum P(E_{i_1} \cap \cdots \cap E_{i_N})$$

$$+ (-1)^{N+1} P(E_1 \cap \cdots \cap E_N) \quad - \quad P\left(\bigcup_{i=1}^{N} (E_i \cap E_{N+1}) \right)$$

$$= \sum_{i=1}^{N+1} P(E_i) + \cdots + (-1)^{\ell+1} \sum_{i_1<\cdots<i_\ell \le N} \cdots \sum P(E_{i_1} \cap \cdots \cap E_{i_N}) + \cdots$$

$$+ (-1)^{N+1} P(E_1 \cap \cdots \cap E_N) -$$

$$\left[\sum_{j=1}^{N} P(E_j \cap E_{N+1}) + \cdots + (-1)^{\nu+1} \sum_{i_1<\cdots<i_\ell \le N} \cdots \sum P(E_{i_1} \cap \cdots \cap E_{i_N} \cap E_{N+1}) + \cdots \right.$$

$$\left. + (-1)^{N+1} P(E_1 \cap \cdots \cap E_{N+1}) \, . \right]$$

Collect terms. #

<u>Example</u> (5m) Suppose each of the N men at a party throws his hat into the center of the room. The hat are mixed up a returned to the men at random. What is the prob. that none of the men gets his own hat back? <u>$S = ?$</u> <u>Permutations</u>?

Let $E_i = \{ \text{the } i\text{th man selects his own hat} \}$.

$$P(\text{none gets his hat}) = P\left(\bigcap_{i=1}^{N} E_i^c\right) = P\left(\left(\bigcup_{i=1}^{N} E_i\right)^c\right)$$

$$= 1 - P\left(\bigcup_{i=1}^{N} E_i\right).$$

Suppose $i_1 < \cdots < i_\ell \le N$. Then $P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_\ell}) = \frac{(N-\ell)!}{N!}$.

$$\therefore \sum_{i_1<\cdots<i_\ell} \cdots \sum P(E_{i_1} \cap \cdots \cap E_{i_\ell}) = \binom{N}{\ell} \frac{(N-\ell)!}{N!} = \frac{1}{\ell!}.$$

$$\Rightarrow \quad P\left(\bigcup_{i=1}^{N} E_i\right) = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots + (-1)^{N+1}\frac{1}{N!}.$$

$\therefore$

$$P\left(\left(\bigcup_{i=1}^{N} E_i\right)^c\right) = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + \cdots + (-1)^N \frac{1}{N!}.$$

$$= \sum_{j=0}^{N} \frac{(-1)^j}{j!}.$$

Recall, $\sum_{j=0}^{\infty} \frac{\lambda^j}{j!} = e^{\lambda}$. So, $P\left(\left(\bigcup_{i=1}^{N} E_i\right)^c\right) \to 1/e \; (\simeq 0.37)$.

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Technical Aside:

The 1 missing axiom of probability is countable additivity:

$\oplus$ If $E_1, E_2, \ldots \subseteq \Omega$ are disjoint (pairwise), then
$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i).$$

This is equivalent to: If $E_1 \supseteq E_2 \supseteq \cdots$ $\bigcap_{i=1}^{\infty} E_i = \phi$, then $P(E_N) \downarrow 0$ as $N \uparrow \infty$. (Continuity from above.)