Two Physical Descriptions of Poisson pdf's

1) The law of rare events.

**Theorem:** If \( X_n : \text{Bin}(n, \frac{\lambda}{n}) \) for a fixed \( \lambda > 0 \), then for all \( k = 0, 1, \ldots \),
\[
\lim_{n \to \infty} P(X_n = k) = e^{-\lambda} \frac{\lambda^k}{k!}.
\]

**Application:** Proportion of a certain property (e.g., lottery, disease, ...)
\[ \frac{1.5}{10^4} \text{ in a random sample of size } 10,000, \]
\[
P[\text{2 people have this property}] \approx \frac{e^{-1.5} \cdot (1.5)^2}{2!} = 0.251024.
\]

The exact pdf is
\[
\binom{10000}{2} \left( \frac{1.5}{10000} \right)^2 \left( 1 - \frac{1.5}{10000} \right)^{9998} = 0.2510434.
\]

**Lemma:** As \( n \to \infty \), \( (1 + \frac{x}{n})^n \to e^x \), \( \forall x \in \mathbb{R} \).

\[
\text{Def } \ln \left( 1 + \frac{x}{n} \right)^n = n \ln \left( 1 + \frac{x}{n} \right).
\]

\[
\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots
\]

\[
\ln \left( 1 + \frac{x}{n} \right)^n = n \left[ \frac{x}{n} - \frac{x^2}{2n^2} + \frac{x^3}{3n^3} - \ldots \right] = x - \frac{x^2}{2n} + \frac{x^3}{3n^2} - \ldots
\]

Want to say \( \frac{1}{n} \cdots \to 0 \). Why and how? [Taylor w/ remainder.]
Pf of Thm  \[ \forall \lambda > 0, \forall n \geq \lambda \]

\[ P \{ X_n = k \} = \frac{m!}{(m-k)! \cdot k!} \cdot \frac{2^k}{n^k} (1 - \frac{\lambda}{n})^{n-k} \]

\[ \begin{align*}
  a) \quad (1 - \frac{\lambda}{n})^{n-k} &= (1 - \frac{\lambda}{n})^n \left(1 - \frac{\lambda}{n}\right)^{-k} \\
  &= e^{-\lambda} \\
  b) \quad \frac{m!}{(m-k)! \cdot n^k} &= \frac{m(m-1) \cdots (m-k+1)}{n^k} \\
  &= \left(\frac{m-1}{n}\right) \left(\frac{m-2}{n}\right) \cdots \left(\frac{n-k+1}{n}\right) \\
  \rightarrow 1. \quad \# \end{align*} \]

Colorful examples: (some taken from p.152)

- The \# of misprints on page (or a few pages) of a book
- The \# of people, in our community, who live 100+ years
- The \# of wrong telephone \#s dialed in a day
- The \# of firearm-related deaths in a day in the US.
- The \# of lottery winners in a given township.

2) Rare arrivals process \hspace{1cm} (Somewhat informal)

"Events" occur as time goes by, that are "rare" in the following sense:

i) In a time-interval of length \( b \), the \( \lambda \)
   of getting exactly 1 occurrence is \( \sim \lambda b \)
   for a fixed \( \lambda b \).

ii) The \( \lambda \) of getting 2 or more time is \( \ll \lambda b \).
iii) The # of occurrences in all disjoint time-intervals are independent.

\[ N(t) := \# \text{ of events that occur by } t \text{. Then,} \]

\[ P \{ N(t) = k \} =? \]

\[ 0 \leq \frac{t}{n} \leq \frac{2t}{n} \ldots \leq t = \frac{mt}{n} \]

Then \[ P \{ N(t) = k \} = P \left[ k \text{ of these } n \text{ intervals have exactly } 1 \text{ exact occurrence among the other } n-k \text{ do not have any} \right. \]

\[ + P \left[ N(t) = k \text{ and at least } 1 \text{ of the } n \text{ subsets} \right] \]

\[ \text{has } 2 \text{ or more occurrences} \]

\[ : = T_1 + T_2 \]

\[ T_2 \leq P \left[ \text{at least } 1 \text{ has } 2 \text{ or more} \right] \]

\[ \leq P \left[ 1^{st} \text{ has } 2 \text{ or more} \right] + P \left[ 2^{nd} \text{ has } 2 \text{ or more} \right] + \ldots + P \left[ n^{th} \text{ has } 2 \text{ or more} \right] \]

\[
\leq \sum_{i=2}^{n-1} \frac{e^{-\lambda t} \lambda^i t^i}{i!}
\]

\[
\ll n \frac{t}{n} = t
\]

\[ \text{i.e., } T_2 \to 0 \text{ as } n \to \infty. \]

"Also, \[ T_1 = P \left[ \text{Bin} \left( n, \frac{t}{n} \right) = k \right] \xrightarrow{n \to \infty} \frac{e^{-\lambda t} \lambda^k}{k!} , \text{ by Law of Rare Events.} \]

\[ P \left[ N(t) = k \right] \text{ does not depend on } n. \text{ So} \]

\[ P \left[ N(t) = k \right] = \frac{e^{-\lambda t} \lambda^k}{k!} ! \]
Colorful Examples (some from p.56)

1) # of earthquakes during a given period
2) # of electrons emitted from a heated cathode for a given period.
3) # of deaths, in a given period, of the policyholders of a given life-insurance company
4) # of messages on a given bulletin board, in a given period.

One more approximation theorem for Binomials (de Moivre, Laplace)

Then

If \( p \) is fixed, and \( X_n \sim \text{Bin}(n,p) \), then

\[
\mathbb{P}\left\{ \frac{X_n - np}{\sqrt{np(1-p)}} \leq x \right\} \xrightarrow{n \to \infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} \, du.
\]

Proof is too hard for now. Much more is done later on.