Recall: given a r.v. X w/ pmf p,

$$E[g(X)] = \sum_y g(y)\, p(y).$$

Ex. Find $E(X^2)$ if X is Poisson(λ). $p(k) = e^{-\lambda}\frac{\lambda^k}{k!}$, $k = 0, 1, \dots$.

$$E[X^2] = \sum x^2 p(x) = \sum_{k=0}^{\infty} k^2 e^{-\lambda}\frac{\lambda^k}{k!}$$

$$= \sum_{k=1}^{\infty} k^2 e^{-\lambda}\frac{\lambda^k}{k!} = \sum_{k=1}^{\infty} k\, e^{-\lambda}\frac{\lambda^k}{(k-1)!}$$

$$= \sum_{k=1}^{\infty} (k-1)\, e^{-\lambda}\frac{\lambda^k}{(k-1)!} + \sum_{k=1}^{\infty} \frac{e^{-\lambda}\lambda^k}{(k-1)!}$$

$$= \sum_{k=2}^{\infty} \frac{e^{-\lambda}\lambda^k}{(k-2)!} + \lambda\sum_{k=0}^{\infty}\frac{e^{-\lambda}\lambda^k}{k!}$$

$$= \lambda^2 + \lambda = [E(X)]^2 + E(X).$$

Ex. $X = \begin{cases} 0 & \frac{1}{6} \\ 1 & \frac{2}{6} \\ 2 & \frac{4}{6} \end{cases} \Rightarrow EX^n = \frac{2}{6} + 2^n \cdot \frac{4}{6}. \quad \forall n \geq \dots$

Def^n $\quad Var(X) := E[(X-\mu)^2]$ if $\mu := E(X)$.

Computational Lemma. $Var(X) = E(X^2) - [E(X)]^2 = E(X^2) - \mu^2$.

Pf. $(X-\mu)^2 = X^2 + \mu^2 - 2\mu X. := g(X)$

$$\therefore\ E(X-\mu)^2 = \sum (x^2 + \mu^2 - 2\mu x)\, p(x)$$

$$= \sum x^2 p(x) + \mu^2 \sum p(x) - 2\mu \sum x p(x)$$

$$= E(X^2) + \mu^2 - 2\mu^2 = E(X^2) - \mu^2. \quad \#$$

Cor (Cauchy - Schwarz Inequality) $|E(X)| \leq \sqrt{E(X^2)}$.

Lemma. $a, b \in \mathbb{R} \Rightarrow \quad Var(aX+b) = a^2 Var(X)$.

Pf. $E(aX+b)^2 = E(\underbrace{a^2X^2 + 2abX + b^2}_{g(X)})$

$$= \sum_x (a^2x^2 + 2abx + b^2)\, p(x)$$

$$= \sum_x a^2x^2 p(x) + 2ab \sum_x x\, p(x) + b^2 \sum p(x)$$

$$= a^2 E(X^2) + 2ab E(X) + b^2. \qquad \text{EQ1}$$

$$[E(aX+b)]^2 = [aEX + b]^2$$

$$= a^2 (EX)^2 + b^2 + 2ab EX. \qquad \text{EQ2}$$

$$\Rightarrow \quad Var(aX+b) = \text{EQ1} - \text{EQ2}$$

$$= a^2 Var(X). \quad \#$$

Def$^{\underline{n}}$ **Std Dev.** $SD(X) := \sqrt{Var(X)}$.

[In particular, $SD(aX+b) = |a|\, SD(X)$.]

Ex X : Bin(n,p); $\mu = np$, $E(X^2) = n^2p^2 + np(1-p)$.

$\Rightarrow$

$Var(X) = np(1-p)$, and $SD(X) = \sqrt{np(1-p)}$.

Ex X : Poisson(λ) $\mu = \lambda$, $EX^2 = \lambda^2 + \lambda$.

$\Rightarrow$

$Var(X) = \lambda$, and $SD(X) = \sqrt{\lambda}$.

Ex. X : Geometric (p) $\mu = 1/p$ by appeal to neg. bin. Here's a different route: $p(k) = p(1-p)^{k-1}$, $k = 1, 2, \dots$

$$E(X) = \sum_{k=1}^{\infty} k\,p\,(1-p)^{k-1} = p \sum_{k=1}^{\infty} k(1-p)^{k-1}.$$

$$\rightarrow \sum_{k=1}^{\infty} k r^{k-1} = \frac{d}{dr}\left(\sum_{k=0}^{\infty} r^k\right) \qquad (0<r<1)$$

$$= \frac{d}{dr}\left(\frac{1}{1-r}\right) = \frac{+1}{(1-r)^2}$$

$\therefore$

$$E(X) = p \Big/ \big(1-(1-p)\big)^2 = 1/p.$$

$$E(X^2) = \sum_{k=1}^{\infty} k^2 p (1-p)^{k-1} = \sum_{k=1}^{\infty} k(k-1)\,p(1-p)^{k-2}(1-p) + \sum_{k=1}^{\infty} kp(1-p)^{k-1}$$

$$= \underbrace{p(1-p)\sum_{k=1}^{\infty} k(k-1)(1-p)^{k-2}}_{?} + \underbrace{\sum_{k=1}^{\infty} kp(1-p)^{k-1}}_{\mu = 1/p}$$

$$\sum_{k=1}^{\infty} k(k-1)(1-p)^{k-2} = \frac{d^2}{dr^2}(1-r)^{-1} = 2(1-r)^{-3}.$$

$$\Rightarrow \quad E(X^2) = p(1-p)\cdot\frac{2}{p^3} + \frac{1}{p} = \frac{2}{p^2} - \frac{2}{p} + \frac{1}{p} = \frac{2}{p^2} - \frac{1}{p}.$$

$$Var(X) = \frac{1}{p^2} - \frac{1}{p} = \frac{1}{p}\left(1 - \frac{1}{p}\right).$$

$$SD(X) = \sqrt{\frac{1}{p}\left(1-\frac{1}{p}\right)}.$$

Ex – (Variant #32 p. 183) $S = \{s_1 \cdots s_n\}$ s_i distinct.

Choose a subset of S at random ; X = # of selected set's elements.

[Q.32 asks for X to be $\neq \emptyset$]

\# of subsets of $S = 2^n = \sum_{j=0}^{n} \binom{n}{j}$

\# of subsets of size $k = \binom{n}{k}$.

$$\therefore \quad P(X=k) = \binom{n}{k} \Big/ 2^n.$$

$$E(X) = 2^{-n} \sum_{k=0}^{n} k \binom{n}{k} = 2^{-n} \sum_{k=1}^{n} k \frac{n!}{k!\,(n-k)!}$$

$$= 2^{-n} \sum_{k=1}^{n} \frac{n!}{(k-1)!\,(n-k)!}$$

$$= 2^{-n} n \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!\,((n-1)-(k-1))!}$$

$$= 2^{-n} n \sum_{k=0}^{n-1} \binom{n-1}{k} = 2^{-n} n\, 2^{n-1} = \frac{n}{2}.$$

$$E(X^2) = 2^{-n}\sum_{k=0}^{n} k^2\binom{n}{k} = 2^{-n}\sum_{k=1}^{n} k(k-1)\binom{n}{k} + \underbrace{2^{-n}\sum_{k=1}^{n} k\binom{n}{k}}_{\mu = \frac{n}{2}}$$

$$= \frac{n}{2} + 2^{-n}\sum_{k=2}^{n} \frac{n!}{(k-2)!\,((n-2)-(k-2))!}$$

$$= \frac{n}{2} + 2^{-n}\, n(n-1)\sum_{k=2}^{n}\binom{n-2}{k-2}$$

$$= \frac{n}{2} + 2^{-n}\, n(n-1)\sum_{j=0}^{n-2}\binom{n-2}{j}$$

$$= \frac{n}{2} + 2^{-n}\, n(n-1)\, 2^{n-2}$$

$$= \frac{n}{2} + \frac{n^2-n}{4} = \frac{n^2}{4} + \frac{n}{4}.$$

$$\therefore \operatorname{Var}(X) = \frac{n}{4}, \qquad SD(X) = \frac{\sqrt{n}}{2}.$$