Problems:

1. A man and a woman agree to meet at a certain location at about 12:30 p.m. If the man arrives at a time uniformly distributed between 12:15 and 12:45, and if the woman independently arrives at a time uniformly distributed between 12:00 and 1 p.m., then find the probability that the first to arrive waits longer than 7 minutes. What is the probability that the man arrives first?

Solution: Let $M$ denote the number of minutes the man arrives after 12:00. Therefore,

$$f_M(m) = \begin{cases} 
\frac{1}{30}, & \text{if } 15 \leq m \leq 45, \\
0, & \text{otherwise}.
\end{cases}$$

Similarly, define $W$ to be the number of minutes the woman arrives after 12:00. Thus,

$$f_W(w) = \begin{cases} 
\frac{1}{60}, & \text{if } 0 \leq w \leq 60, \\
0, & \text{otherwise}.
\end{cases}$$

By independence,

$$f_{M,W}(m, w) = \begin{cases} 
\frac{1}{1800}, & \text{if } 10 \leq m \leq 45 \text{ and } 0 \leq w \leq 60, \\
0, & \text{otherwise}.
\end{cases}$$

We are interested in finding $P\{M < W - 7 \text{ or } W < M - 7\}$. Now $f_{M,W}$ is zero off the rectangle $R := \{(m, w) : 15 \leq m \leq 45, 0 \leq w \leq 60\}$. Let $A$ denote the area, in $R$, that is bounded between the two lines $w = m + 7$ and $w = m - 7$. Then,

$$P\{M < W - 7 \text{ or } W < M - 7\} = 1 - \frac{\text{Area}(A)}{\text{Area}(R)} = 1 - \frac{\text{Area}(A)}{1800}.$$ 

But $A$ is a parallelopiped; so the area is base $\times$ height. By the Pythagorean rule, base($A$) = $\sqrt{900 + 900} = 30\sqrt{2}$. Also, the height is described by $\sin(45^\circ) = \text{height}(A)/14$. Because $\sin(45^\circ) = 1/\sqrt{2}$, this yields height($A$) = $14/\sqrt{2}$. Therefore, area($A$) = $30 \times 14 = 420$, and

$$P\{M < W - 7 \text{ or } W < M - 7\} = 1 - \frac{420}{1800} = \frac{23}{30}.$$ 

2. When a current $I$ (in amperes) flows through a resistance $R$ (in ohms), the power generated is given by $W = I^2R$ (in watts). Suppose that $I$ and $R$ are independent random variables with densities

$$f_I(x) = 6x(1-x) \quad 0 \leq x \leq 1,$$

$$f_R(x) = 2x \quad 0 \leq x \leq 1.$$
Then find the density of $W$. Use this to compute $EW$. 

**Solution:** We have 

$$f_{I,R}(x,y) = \begin{cases} 12xy(1-x), & \text{if } 0 \leq x, y \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, for all $0 \leq a \leq 1$,

$$F_W(a) = P\{W \leq a\} = P\{I^2R \leq a\} = \int \int_{\mathcal{R}} f_{I,R}(x,y) \, dx \, dy,$$

where $\mathcal{R}$ is the region inside the square $(0 \leq x, y \leq 1)$ that is under the curve $y = a/x^2$. This region is a union of a rectangle and one bounded by a hyperbola. Therefore, for all $0 \leq a \leq 1$,

$$F_W(a) = \int_0^a \int_0^1 12xy(1-x) \, dx \, dy + \int_1^a \int_0^{\sqrt{a/y}} 12xy(1-x) \, dx \, dy$$

$$= a^2 + 6a(1-a) - 8a^{3/2} \left(1 - a^{1/2}\right).$$

Because $W \geq 0$, $F_W(a) = 0$ if $a < 0$. Also, $F_W(a) = 1$ if $a > 1$. Therefore,

$$f_W(a) = F_W'(a) = \begin{cases} 6a - 12\sqrt{a} + 6, & \text{if } 0 \leq a \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Consequently,

$$EW = \int_0^1 \left(6a^2 - 12a^{3/2} + 6a\right) \, da = -3.$$

**3.** Suppose that 3 balls are chosen without replacement from an urn consisting of 5 white and 8 red balls. Let $X_i$ equal 1 if the $i$th ball selected is white, and let $X_i$ equal 0 otherwise. Compute the conditional probability mass function of $X_1$ given that $X_2 = 1$. Use this to find $E[X_1 \mid X_2 = 1]$.

**Solution:** Let $W_i$ denote the event that we drew white on the $i$th draw. Then,

$$P\left(X_1 = 1 \mid X_2 = 1\right) = P\left(W_1 \mid W_2\right) = P\left(W_2 \mid W_1\right) \frac{P(W_1)}{P(W_2)} = P\left(W_2 \mid W_1\right) = \frac{1}{3}.$$ 

Also, $P(X_1 = 0 \mid X_2 = 1) = 2/3$. Thus,

$$E\left[X_1 \mid X_2 = 1\right] = \left(1 \times P\left(X_1 = 1 \mid X_2 = 1\right) \right) + \left(0 \times P\left(X_1 = 0 \mid X_2 = 1\right) \right) = \frac{1}{3}.$$

**4.** The density of $(X,Y)$ is

$$f(x,y) = \begin{cases} xe^{-x(y+1)}, & \text{if } x > 0 \text{ and } y > 0, \\ 0, & \text{otherwise.} \end{cases}$$
(a)  Find the conditional density of $X$, given $Y = y$. Use this to compute $E[X \mid Y = y]$.

**Solution:** First of all the density of $Y$ at $y > 0$ is

$$f_Y(y) = \int_0^\infty xe^{-x(y+1)} \, dx = \frac{1}{(1 + y)^2}.$$  

If $y \leq 0$ then $f_Y(y) = 0$. As a result, we have

$$f_{X \mid Y}(x \mid y) = x(y + 1)^2 e^{-x(y+1)}, \quad x > 0,$$

and $f_{X \mid Y}(x \mid y) = 0$ for $x \leq 0$. Thus,

$$E[X \mid Y = y] = \int_0^\infty x^2(y + 1)^2 e^{-x(y+1)} \, dx = \frac{2}{1 + y}.$$

(b)  Find the conditional density of $Y$, given $X = x$. Use this conditional density to compute $E[Y \mid X = x]$.

**Solution:** First of all the density of $X$ at $x > 0$ is

$$f_X(x) = \int_0^\infty xe^{-x(y+1)} \, dy = e^{-x}.$$  

If $x \leq 0$ then $f_X(x) = 0$. As a result, we have

$$f_{Y \mid X}(y \mid x) = xe^{-xy}, \quad y > 0,$$

and $f_{Y \mid X}(y \mid x) = 0$ for $y \leq 0$. Thus,

$$E[Y \mid X = x] = \int_0^\infty xy e^{-xy} \, dy = \frac{1}{x}.$$

(c)  Find the density of $Z = XY$. Use this to compute $E[Z]$ and $\text{Var}(Z)$.

**Solution:** $F_Z(a) = 0$ if $a < 0$. But if $a \geq 0$ then

$$F_Z(a) = P\{XY \leq a\} = \int_0^\infty \int_0^{a/y} xe^{-x(y+1)} \, dx \, dy.$$  

Therefore,

$$f_Z(a) = \frac{dF_Z(a)}{da} = \int_0^\infty \frac{d}{da} \left( \int_0^{a/y} xe^{-x(y+1)} \, dx \right) \, dy.$$  

By the fundamental theorem of calculus,

$$f_Z(a) = \int_0^\infty \frac{a}{y^2} e^{-a(y+1)/y} \, dy = \int_0^\infty ay^{-2} e^{-a(1+y^{-1})} \, dy.$$
Set $z := y^{-1}$ to find that $dz = -y^{-2}dy$, and so

$$f_Z(a) = \int_0^\infty ae^{-a(1+z)}\,dz = e^{-a}, \quad a > 0.$$ 

And $f_Z(a) = 0$ if $a \leq 0$. Therefore, $Z$ is exponential with mean one. So $EZ = 1$ and $\text{Var}(Z) = 1$.

Theoretical Problems:

1. Suppose $X$ is exponentially distributed with parameter $\lambda > 0$. Find

$$P\{[X] = n, \ X - [X] \leq x\},$$

where $[a]$ denotes the largest integer $\leq a$. Are $X$ and $X - [X]$ independent?

**Solution:** If $x$ is a positive number and $n$ is a positive integer, then $\lfloor x \rfloor = n$ is synonymous to $n \leq x \leq n + 1$. Therefore,

$$P\{[X] = n, \ X - [X] \leq x\} = P\{n \leq X \leq n + 1, \ X \leq n + x\}.$$

There are three cases to consider:

(i) If $x \geq 1$ then

$$P\{[X] = n, \ X - [X] \leq x\} = P\{n \leq X \leq n + 1\} = \int_n^{n+1} e^{-z}\,dz = e^{-n} - e^{-n-1}.$$

(ii) If $0 \leq x \leq 1$, then

$$P\{[X] = n, \ X - [X] \leq x\} = P\{n \leq X \leq n + x\} = \int_n^{n+x} e^{-z}\,dz = e^{-n} - e^{-n-x}.$$

(iii) If $x < 0$, then the probability in question is zero.

No, $X$ and $X - [X]$ are not independent, as can be seen from the above discussion.

2. Suppose $X$ and $Y$ are independent, standard normal random variables. Prove that $Z := X/Y$ has the Cauchy density. That is, prove that the density of $Z$ is

$$f_Z(a) = \frac{1}{\pi(1 + a^2)}, \quad -\infty < a < \infty.$$ 

**Solution:** We start, as before, and compute $F_Z$ first, and then differentiate. You will need to draw the region of integration in order to follow this discussion.

We begin with the observation that

$$F_Z(a) = P\left\{\frac{X}{Y} \leq a\right\} = P\{X \leq aY, \ Y \geq 0\} + P\{X \geq aY, \ Y < 0\}. \quad (\text{eq}0)$$
Now, \[
P\{X \leq aY, Y \geq 0\} = \int_0^\infty \left( \int_{-\infty}^{ay} \frac{e^{-x^2/2}}{\sqrt{2\pi}} \, dx \right) \frac{e^{-y^2/2}}{\sqrt{2\pi}} \, dy.
\]
By the fundamental theorem of calculus, \(\frac{d}{da} \left( \int_{-\infty}^{ay} e^{-y^2/2} \, dy \right) = ye^{-a^2y^2/2}\). Therefore,
\[
\frac{d}{da} P\{X \leq aY, Y \geq 0\} = \int_0^\infty \frac{ye^{-y^2a^2/2} e^{-y^2/2}}{\sqrt{2\pi}} \, dy = \frac{1}{2\pi} \int_0^\infty ye^{-(1+a^2)y^2/2} \, dy
\]
\[
= \frac{1}{2\pi(1+a^2)} \int_0^\infty e^{-z} \, dz \quad [z := y^2(1+a^2)/2] \quad (eq.1)
\]
Very similar computations show that
\[
\frac{d}{da} P\{X \geq aY, Y < 0\} = \frac{1}{2\pi(1+a^2)}.
\]
Combine (eq.0), (eq.1), and (eq.2) to deduce the result.

3. \textit{Suppose }X \textit{ is a standard normal random variable. Compute the density of } Y = X^2. 
\textbf{Solution:} \textit{Once more, we start with the distribution function: If } a < 0 \textit{ then } F_Y(a) = 0 \textit{ because } Y \geq 0. \textit{ Else if } a \geq 0 \textit{ then}
\[
F_Y(a) = P\{-\sqrt{a} \leq X \leq \sqrt{a}\} = \Phi(\sqrt{a}) - \Phi\left(-\sqrt{a}\right).
\]
But if \(\alpha \geq 0\) then \(\Phi(-\alpha) = 1 - \Phi(\alpha)\), by symmetry. Hence,
\[
F_Y(a) = 2\Phi(\sqrt{a}) - 1.
\]
Differentiate to find that
\[
f_Y(a) = F_Y'(a) = 2\Phi'(\sqrt{a}) \times \frac{1}{2\sqrt{a}} = \frac{e^{-a/2}}{\sqrt{2\pi a}},
\]
if \(a > 0\). Else, \(f_Y(a) = 0\).