Solutions to Assignment #8
Math 501–1, Spring 2006
University of Utah

Problems:

1. *A man and a woman agree to meet at a certain location at about 12:30 p.m. If the man arrives at a time uniformly distributed between 12:15 and 12:45, and if the woman independently arrives at a time uniformly distributed between 12:00 and 1 p.m., then find the probability that the first to arrive waits longer than 7 minutes. What is the probability that the man arrives first?*

Solution: Let M denote the number of minutes the man arrives after 12:00. Therefore,

$$f_M(m) = \begin{cases} \frac{1}{30}, & \text{if } 15 \le m \le 45, \\ 0, & \text{otherwise.} \end{cases}$$

Similarly, define W to be the number of minutes the woman arrives after 12:00. Thus,

$$f_W(w) = \begin{cases} \frac{1}{60}, & \text{if } 0 \le w \le 60, \\ 0, & \text{otherwise.} \end{cases}.$$

By independence,

$$f_{M,W}(m\,,w) = \begin{cases} \frac{1}{1800}, & \text{if } 10 \le m \le 45 \text{ and } 0 \le w \le 60, \\ 0, & \text{otherwise.} \end{cases}$$

We are interested in finding $P\{M < W - 7 \text{ or } W < M - 7\}$. Now $f_{M,W}$ is zero off the rectangle $\mathcal{R} := \{(m\,,w) :\ 15 \le m \le 45\ ,\ 0 \le w \le 60\}$. Let $\mathcal{A}$ denote the area, in $\mathcal{R}$, that is bounded between the two lines $w = m + 7$ and $w = m - 7$. Then,

$$P\{M < W - 7 \text{ or } W < M - 7\} = 1 - \frac{\text{Area}(\mathcal{A})}{\text{Area}(\mathcal{R})} = 1 - \frac{\text{Area}(\mathcal{A})}{1800}.$$

But $\mathcal{A}$ is a parallelopiped; so the area is base $\times$ height. By the Pythagorean rule, base$(\mathcal{A}) = \sqrt{900 + 900} = 30\sqrt{2}$. Also, the height is described by $\sin(45°) =$ height$(\mathcal{A})/14$. Because $\sin(45°) = 1/\sqrt{2}$, this yields height$(\mathcal{A}) = 14/\sqrt{2}$. Therefore, area$(\mathcal{A}) = 30 \times 14 = 420$, and

$$P\{M < W - 7 \text{ or } W < M - 7\} = 1 - \frac{420}{1800} = \frac{23}{30}.$$

2. *When a current I (in amperes) flows through a resistance R (in ohms), the power generated is given by $W = I^2 R$ (in watts). Suppose that I and R are independent random variables with densities*

$$f_I(x) = 6x(1 - x) \qquad 0 \le x \le 1,$$
$$f_R(x) = 2x \qquad 0 \le x \le 1.$$

Then find the density of W. Use this to compute EW.

Solution: We have

$$f_{I,R}(x\,,y)=\begin{cases}12xy(1-x), & \text{if } 0\le x,y\le 1,\\ 0, & \text{otherwise.}\end{cases}$$

Thus, for all $0\le a\le 1$,

$$\begin{aligned}F_W(a)&=P\{W\le a\}=P\left\{I^2R\le a\right\}\\&=\iint\limits_{\mathcal{R}} f_{I,R}(x\,,y)\,dx\,dy,\end{aligned}$$

where $\mathcal{R}$ is the region inside the square $(0\le x,y\le 1)$ that is under the curve $y=a/x^2$. This region is a union of a rectangle and one bounded by a hyperbola. Therefore, for all $0\le a\le 1$,

$$\begin{aligned}F_W(a)&=\int_0^a\int_0^1 12xy(1-x)\,dx\,dy+\int_a^1\int_0^{\sqrt{a/y}}12xy(1-x)\,dx\,dy\\&=a^2+6a(1-a)-8a^{3/2}\left(1-a^{1/2}\right).\end{aligned}$$

Because $W\ge 0$, $F_W(a)=0$ if $a<0$. Also, $F_W(a)=1$ if $a>1$. Therefore,

$$f_W(a)=F_W'(a)=\begin{cases}6a-12\sqrt{a}+6, & \text{if } 0\le a\le 1,\\ 0, & \text{otherwise.}\end{cases}$$

Consequently,

$$EW=\int_0^1\left(6a^2-12a^{3/2}+6a\right)\,da=-3.$$

3. *Suppose that 3 balls are chosen without replacement from an urn consisting of 5 white and 8 red balls. Let X_i equal 1 if the ith ball selected is white, and let X_i equal 0 otherwise. Compute the conditional probability mass function of X_1 given that $X_2=1$. Use this to find $E[X_1\,|\,X_2=1]$.*

Solution: Let W_i denote the event that we drew white on the ith draw. Then,

$$P\left(X_1=1\,|\,X_2=1\right)=P\left(W_1\,|\,W_2\right)=P\left(W_2\,|\,W_1\right)\frac{P(W_1)}{P(W_2)}=P\left(W_2\,|\,W_1\right)=\frac{1}{3}.$$

Also, $P(X_1=0\,|\,X_2=1)=2/3$. Thus,

$$E\left[X_1\,|\,X_2=1\right]=\Big(1\times P\left(X_1=1\,|\,X_2=1\right)\Big)+\Big(0\times P\left(X_1=0\,|\,X_2=1\right)\Big)=\frac{1}{3}.$$

4. *The density of $(X\,,Y)$ is*

$$f(x\,,y)=\begin{cases}xe^{-x(y+1)}, & \text{if } x>0 \text{ and } y>0,\\ 0, & \text{otherwise.}\end{cases}$$

(a) *Find the conditional density of X, given $Y = y$. Use this to compute $E[X \mid Y = y]$.*

Solution: First of all the density of Y at $y > 0$ is

$$f_Y(y) = \int_0^\infty x e^{-x(y+1)}\, dx = \frac{1}{(1+y)^2}.$$

If $y \le 0$ then $f_Y(y) = 0$. As a result, we have

$$f_{X|Y}(x \mid y) = x(y+1)^2 e^{-x(y+1)}, \qquad x > 0,$$

and $f_{X|Y}(x \mid y) = 0$ for $x \le 0$. Thus,

$$E[X \mid Y = y] = \int_0^\infty x^2 (y+1)^2 e^{-x(y+1)}\, dx = \frac{2}{1+y}.$$

(b) *Find the conditional density of Y, given $X = x$. Use this conditional density to compute $E[Y \mid X = x]$.*

Solution: First of all the density of X at $x > 0$ is

$$f_X(x) = \int_0^\infty x e^{-x(y+1)}\, dy = e^{-x}.$$

If $x \le 0$ then $f_X(x) = 0$. As a result, we have

$$f_{Y|X}(y \mid x) = x e^{-xy}, \qquad y > 0,$$

and $f_{Y|X}(y \mid x) = 0$ for $y \le 0$. Thus,

$$E[Y \mid X = x] = \int_0^\infty x y e^{-xy}\, dy = \frac{1}{x}.$$

(c) *Find the density of $Z = XY$. Use this to compute $E[Z]$ and $\mathrm{Var}(Z)$.*

Solution: $F_Z(a) = 0$ if $a < 0$. But if $a \ge 0$ then

$$F_Z(a) = P\{XY \le a\} = \int_0^\infty \int_0^{a/y} x e^{-x(y+1)}\, dx\, dy.$$

Therefore,

$$f_Z(a) = \frac{dF_Z(a)}{da} = \int_0^\infty \frac{d}{da} \left(\int_0^{a/y} x e^{-x(y+1)}\, dx \right) dy.$$

By the fundamental theorem of calculus,

$$f_Z(a) = \int_0^\infty \frac{a}{y^2} e^{-a(y+1)/y}\, dy = \int_0^\infty a y^{-2} e^{-a(1+y^{-1})}\, dy.$$

Set $z := y^{-1}$ to find that $dz = -y^{-2}dy$, and so

$$f_Z(a) = \int_0^\infty a e^{-a(1+z)}\,dz = e^{-a}, \qquad a > 0.$$

And $f_Z(a) = 0$ if $a \le 0$. Therefore, Z is exponential with mean one. So $EZ = 1$ and $\operatorname{Var}(Z) = 1$.

Theoretical Problems:

1. *Suppose X is exponentially distributed with paramater $\lambda > 0$. Find*

$$P\{[X] = n\ ,\ X - [X] \le x\},$$

where $[a]$ denotes the largest integer $\le a$. Are X and $X - [X]$ independent?

Solution: If x is a positive number and n is a positive integer, then "$[x] = n$" is synonymous to "$n \le x \le n+1$." Therefore,

$$P\{[X] = n\ ,\ X - [X] \le x\} = P\{n \le X \le n+1\ ,\ X \le n + x\}.$$

There are three cases to consider: **(i)** If $x \ge 1$ then

$$\begin{aligned} P\{[X] = n\ ,\ X - [X] \le x\} = P\{n \le X \le n+1\} &= \int_n^{n+1} e^{-z}\,dz \\ &= e^{-n} - e^{-n-1}. \end{aligned}$$

(ii) If $0 \le x \le 1$, then

$$\begin{aligned} P\{[X] = n\ ,\ X - [X] \le x\} = P\{n \le X \le n+x\} &= \int_n^{n+x} e^{-z}\,dz \\ &= e^{-n} - e^{-n-x}. \end{aligned}$$

(iii) If $x < 0$, then the probability in question is zero.

No, X and $X - [X]$ are not independent, as can be seen from the above discussion.

2. *Suppose X and Y are independent, standard normal random variables. Prove that $Z := X/Y$ has the Cauchy density. That is, prove that the density of Z is*

$$f_Z(a) = \frac{1}{\pi(1+a^2)}, \qquad -\infty < a < \infty.$$

Solution: We start, as before, and compute F_Z first, and then differentiate. You will need to draw the region of integration in order to follow this discussion.

We begin with the observation that

$$F_Z(a) = P\left\{\frac{X}{Y} \le a\right\} = P\{X \le aY\ ,\ Y \ge 0\} + P\{X \ge aY\ ,\ Y < 0\}. \qquad \text{(eq.0)}$$

Now,

$$P\{X \le aY\ ,\ Y \ge 0\} = \int_0^\infty \left(\int_{-\infty}^{ay} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\, dx \right) \frac{e^{-y^2/2}}{\sqrt{2\pi}}\, dy.$$

By the fundamental theorem of calculus, $(d/da)\big(\int_{-\infty}^{ay} e^{-y^2/2}\, dy\big) = ye^{-a^2y^2/2}$. Therefore,

$$\begin{aligned} \frac{d}{da} P\{X \le aY\ ,\ Y \ge 0\} &= \int_0^\infty \frac{ye^{-y^2a^2/2}}{\sqrt{2\pi}} \frac{e^{-y^2/2}}{\sqrt{2\pi}}\, dy = \frac{1}{2\pi} \int_0^\infty ye^{-(1+a^2)y^2/2}\, dy \\ &= \frac{1}{2\pi(1+a^2)} \int_0^\infty e^{-z}\, dz \qquad \big[z := y^2(1+a^2)/2\big] \qquad \text{(eq.1)} \\ &= \frac{1}{2\pi(1+a^2)}. \end{aligned}$$

Very similar computations show that

$$\frac{d}{da} P\{X \ge aY\ ,\ Y < 0\} = \frac{1}{2\pi(1+a^2)}. \qquad \text{(eq.2)}$$

Combine (eq.0), (eq.1), and (eq.2) to deduce the result.

3. *Suppose X is a standard normal random variable. Compute the density of $Y = X^2$.*

Solution: Once more, we start with the distribution function: If $a < 0$ then $F_Y(a) = 0$ because $Y \ge 0$. Else if $a \ge 0$ then

$$F_Y(a) = P\left\{-\sqrt{a} \le X \le \sqrt{a}\right\} = \Phi\left(\sqrt{a}\right) - \Phi\left(-\sqrt{a}\right).$$

But if $\alpha \ge 0$ then $\Phi(-\alpha) = 1 - \Phi(\alpha)$, by symmetry. Hence,

$$F_Y(a) = 2\Phi\left(\sqrt{a}\right) - 1.$$

Differentiate to find that

$$f_Y(a) = F_Y'(a) = 2\Phi'\left(\sqrt{a}\right) \times \frac{1}{2\sqrt{a}} = \frac{e^{-a/2}}{\sqrt{2\pi a}},$$

if $a > 0$. Else, $f_Y(a) = 0$.