Solutions to Assignment #6
Math 501–1, Spring 2006
University of Utah

Problems:

1. *Let X be a random variable with density function*

$$f(x) = \begin{cases} c(1-x^2) & \text{if } -1 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) *What is c?*

Solution: $1 = c\int_{-1}^{1}(1-x^2)\,dx = 2c - c\int_{-1}^{1} x^2\,dx = 2c - (2c/3) = 4c/3$, so $c = 3/4$.

(b) *Compute the distribution function F.*

Solution: If $a > 1$ then $F(a) = 1$; if $a < -1$ then $F(a) = 0$. For all a between ± 1, we have

$$F(a) = \frac{3}{4}\int_{-1}^{a}(1-x^2)\,dx = \frac{3}{4}\left[a + 1 - \int_{-1}^{a} x^2\,dx\right] = \frac{3}{4}\left[a + 1 - \frac{1}{3}\left(a^3+1\right)\right].$$

(c) *Calculate $P\{0 < X < 1.5\}$.*

Solution: This is the same $P\{0 < X < 1\} = (3/4)\int_0^1 (1-x^2)\,dx = 1/2$.

(d) *Compute EX and Var$X$.*

Solution: Note that $x(1-x^2)$ is an odd function as x varies over $[-1\,,1]$. Therefore, $EX = 0$. For the variance we first need $E(X^2)$, viz.,

$$\begin{aligned} E[X^2] &= \frac{3}{4}\int_{-1}^{1} x^2(1-x^2)\,dx = \frac{3}{4}\int_{-1}^{1}(x^2 - x^4)\,dx \\ &= \frac{3}{4}\left[\frac{2}{3} - \frac{2}{5}\right] = \frac{1}{5}. \end{aligned}$$

Therefore, $\mathrm{Var}(X) = E[X^2] - |EX|^2 = (1/5)$.

2. *Suppose X is normally distributed with mean $\mu = 10$ and variance $\sigma^2 = 36$. Compute:*

(a) $P\{X > 5\}$.

Solution: Standardize to find that

$$P\{X \le 5\} = \Phi\left(\frac{5-10}{6}\right) = \Phi(-0.8\bar{3}) = 1 - \Phi(0.8\bar{3}) \approx 1 - 0.7967.$$

Therefore, $P\{X > 5\} \approx 0.7967$.

(b) $P\{4 < X < 16\}$.

Solution: Because $P\{X = 16\} = 0$, we have

$$P\{4 < X < 16\} = P\{X \leq 16\} - P\{X \leq 4\} = \Phi\left(\frac{16-10}{6}\right) - \Phi\left(\frac{4-10}{6}\right)$$
$$= \Phi(1) - \Phi(-1) = 2\Phi(1) - 1 \approx (2 \times 0.8413) - 1 = 0.6826.$$

3. *Let X be uniformly distributed on $[0\,,1]$. Then compute $E[X^n]$ for all integers $n \geq 1$. What happens if $n = -1$?*

Solution: $E[X^n] = \int_0^1 x^n\,dx = 1/(n+1)$. If $n = -1$, then this is $\int_0^1 x^{-1}\,dx = \infty$, so $E[1/X] = \infty$.

4. *The density function of X is*

$$f(x) = \begin{cases} a + bx^2 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

We know that $EX = 3/5$. Compute a and b.

Solution: To begin with, $\int_{-\infty}^{\infty} f(x)\,dx = 1$. So,

$$1 = \int_0^1 (a + bx^2)\,dx = a + \frac{b}{3}. \qquad \text{(eq.1)}$$

Also,

$$\frac{3}{5} = \int_0^1 x(a + bx^2)\,dx = \frac{a}{2} + \frac{b}{4}. \qquad \text{(eq.2)}$$

Consider $2 \times$ (eq.2) $-$ (eq.1):

$$\frac{1}{5} = \frac{b}{2} - \frac{b}{3} = \frac{b}{6}.$$

Thus, $b = \frac{6}{5}$. Also, by (eq.1), $a = 1 - \frac{b}{3} = 1 - \frac{2}{5} = \frac{3}{5}$.

Theoretical Problems:

1. *Let X have the exponential(λ) distribution, where $\lambda > 0$ is fixed. That is, we suppose that the density function of X is*

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Prove that for all integers $k \geq 1$,

$$E[X^k] = \frac{k!}{\lambda^k}.$$

Solution: We compute directly to find that

$$E[X^k] = \int_0^\infty x^k \lambda e^{-\lambda x}\, dx$$
$$= \frac{1}{\lambda^k}\int_0^\infty y^k e^{-y}\, dy = \frac{\Gamma(k+1)}{\lambda^k} = \frac{k!}{\lambda^k}.$$

2. *Let X have the exponential(λ) distribution, where $\lambda > 0$ is fixed. Then, compute $P(X > x+y \,|\, X > y)$ for all $x, y > 0$. Use this to prove that for all $x, y > 0$,*

$$P(X > x+y \,|\, X > y) = P\{X > x\}.$$

This property is called "memoryless-ness."

Solution: We compute directly:

$$P\{X > x\} = \int_x^\infty \lambda e^{-\lambda z}\, dz = e^{-\lambda x}, \qquad \text{for all } x > 0.$$

On the other hand,

$$P\{X > x+y \,|\, X > y\} = \frac{P\{X > x+y\ ,\ X > x\}}{P\{X > y\}} = \frac{P\{X > x+y\}}{P\{X > y\}}$$
$$= \frac{e^{-\lambda(x+y)}}{e^{-\lambda y}} = e^{-\lambda x}, \quad \text{for all } x, y > 0.$$