Solutions to Assignment #6
Math 501–1, Spring 2006
University of Utah

Problems:

1. Let $X$ be a random variable with density function

   $f(x) = \begin{cases} 
   c(1 - x^2) & \text{if } -1 < x < 1, \\
   0 & \text{otherwise.} 
   \end{cases}$

   (a) What is $c$?
   
   Solution: $1 = c \int_{-1}^{1} (1 - x^2) \, dx = 2c - c \int_{-1}^{1} x^2 \, dx = 2c - (2c/3) = 4c/3$, so $c = 3/4$.

   (b) Compute the distribution function $F$.
   
   Solution: If $a > 1$ then $F(a) = 1$; if $a < -1$ then $F(a) = 0$. For all $a$ between $\pm 1$, we have

   $$F(a) = \frac{3}{4} \int_{-1}^{a} (1 - x^2) \, dx = \frac{3}{4} \left[ a + 1 - \int_{-1}^{a} x^2 \, dx \right] = \frac{3}{4} \left[ a + 1 - \frac{1}{3} (a^3 + 1) \right].$$

   (c) Calculate $P\{0 < X < 1.5\}$.
   
   Solution: This is the same $P\{0 < X < 1\} = (3/4) \int_{0}^{1} (1 - x^2) \, dx = 1/2$.

   (d) Compute $EX$ and $VarX$.
   
   Solution: Note that $x(1 - x^2)$ is an odd function as $x$ varies over $[-1, 1]$. Therefore, $EX = 0$. For the variance we first need $E(X^2)$, viz.,

   $$E[X^2] = \frac{3}{4} \int_{-1}^{1} x^2(1 - x^2) \, dx = \frac{3}{4} \int_{-1}^{1} (x^2 - x^4) \, dx$$

   $$= \frac{3}{4} \left[ \frac{2}{3} - \frac{2}{5} \right] = \frac{1}{5}.$$

   Therefore, $Var(X) = E[X^2] - |EX|^2 = (1/5)$.

2. Suppose $X$ is normally distributed with mean $\mu = 10$ and variance $\sigma^2 = 36$. Compute:
   
   (a) $P\{X > 5\}$.
   
   Solution: Standardize to find that

   $$P\{X \leq 5\} = \Phi \left( \frac{5 - 10}{6} \right) = \Phi(-0.83) = 1 - \Phi(0.83) \approx 1 - 0.7967.$$

   Therefore, $P\{X > 5\} \approx 0.7967$.

   (b) $P\{4 < X < 16\}$.
Solution: Because $P\{X = 16\} = 0$, we have

\[ P\{4 < X < 16\} = P\{X \leq 16\} - P\{X \leq 4\} = \Phi \left( \frac{16 - 10}{6} \right) - \Phi \left( \frac{4 - 10}{6} \right) = \Phi(1) - \Phi(-1) = 2\Phi(1) - 1 \approx (2 \times 0.8413) - 1 = 0.6826. \]

3. Let $X$ be uniformly distributed on $[0, 1]$. Then compute $E[X^n]$ for all integers $n \geq 1$. What happens if $n = -1$?

Solution: $E[X^n] = \int_0^1 x^n \, dx = 1/(n + 1)$. If $n = -1$, then this is $\int_0^1 x^{-1} \, dx = \infty$, so $E[1/X] = \infty$.

4. The density function of $X$ is

\[ f(x) = \begin{cases} a + bx^2 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise}. \end{cases} \]

We know that $EX = 3/5$. Compute $a$ and $b$.

Solution: To begin with, $\int_{-\infty}^\infty f(x) \, dx = 1$. So,

\[ 1 = \int_0^1 (a + bx^2) \, dx = a + \frac{b}{3}. \quad (\text{eq.1}) \]

Also,

\[ \frac{3}{5} = \int_0^1 x(a + bx^2) \, dx = \frac{a}{2} + \frac{b}{4}. \quad (\text{eq.2}) \]

Consider $2 \times (\text{eq.2}) - (\text{eq.1})$:

\[ \frac{1}{5} = \frac{b}{2} - \frac{b}{3} = \frac{b}{6}. \]

Thus, $b = \frac{6}{5}$. Also, by (eq.1), $a = 1 - \frac{b}{3} = 1 - \frac{2}{5} = \frac{3}{5}$.

Theoretical Problems:

1. Let $X$ have the exponential($\lambda$) distribution, where $\lambda > 0$ is fixed. That is, we suppose that the density function of $X$ is

\[ f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0, \\ 0 & \text{otherwise}. \end{cases} \]

Prove that for all integers $k \geq 1$,

\[ E[X^k] = \frac{k!}{\lambda^k}. \]
Solution: We compute directly to find that

\[
E[X^k] = \int_0^\infty x^k \lambda e^{-\lambda x} \, dx \\
= \frac{1}{\lambda^k} \int_0^\infty y^k e^{-y} \, dy = \frac{\Gamma(k+1)}{\lambda^k} = \frac{k!}{\lambda^k}.
\]

2. Let \( X \) have the exponential(\( \lambda \)) distribution, where \( \lambda > 0 \) is fixed. Then, compute \( P(X > x + y \mid X > y) \) for all \( x, y > 0 \). Use this to prove that for all \( x, y > 0 \),

\[
P(X > x + y \mid X > y) = P\{X > x\}.
\]

This property is called "memoryless-ness."

Solution: We compute directly:

\[
P\{X > x\} = \int_x^\infty \lambda e^{-\lambda z} \, dz = e^{-\lambda x}, \quad \text{for all } x > 0.
\]

On the other hand,

\[
P\{X > x + y \mid X > y\} = \frac{P\{X > x + y , X > x\}}{P\{X > y\}} = \frac{P\{X > x + y\}}{P\{X > y\}}
\]

\[
= \frac{e^{-\lambda(x+y)}}{e^{-\lambda y}} = e^{-\lambda x}, \quad \text{for all } x, y > 0.
\]