Solutions to Problem Assignment #5
Math 501–1, Spring 2006
University of Utah

Problems:

1. *A gambling book recommends the following "winnin strategy" for the game of roulette: It recommends that the gambler bet \$1 on red. If red appears (this has probability 18/38), then the gambler should take her \$1 profit and quit. Else, she should make additional \$1 bets on red on each of the next two spins of the roulette wheel, and then quit. Let X denote the gambler's winnings when she quits.*

(a) *Find $P\{X > 0\}$.*

Solution. Let R_i denote the event, "red on the ith trial." Because $P(R_i) = 18/38$ and $P(R_i^c) = 20/38$,

$$P\{X = 1\} = P(R_1) + P(R_1^c \cap R_2 \cap R_3) = \frac{18}{38} + \left(\frac{20}{38} \cdot \frac{18}{38} \cdot \frac{18}{38}\right) = 0.591\bar{7}.$$

On the other hand,

$$\begin{aligned} P\{X = -1\} &= P(R_1^c \cap R_2 \cap R_3^c) + P(R_1^c \cap R_2^c \cap R_3) \\ &= \left(\frac{20}{38} \cdot \frac{18}{38} \cdot \frac{20}{38}\right) + \left(\frac{20}{38} \cdot \frac{20}{38} \cdot \frac{18}{38}\right) \approx 0.2624, \\ P\{X = -3\} &= P(R_1^c \cap R_2^c \cap R_3^c) = \left(\frac{20}{38}\right)^3 \approx 0.1458. \end{aligned}$$

In particular, $P\{X > 0\} = P\{X = 1\} = 0.591\bar{7}$.

(b) *Are you convinced that the strategy is indeed a "winning" strategy? Explain your answer.*

Solution. One may think at first that this is a winning strategy because $P\{\text{win}\} > 0.5$. But it is the expectation that counts, and we will see next that $EX < 0$. This means that the strategy is a losing strategy.

(c) *Compute EX.*

Solution. We have

$$EX = 0.591\bar{7} - 0.2624 - (3 \times 0.1458) \approx -0.108.$$

Therefore, every time we play we expect to lose about 10 cents.

2. *One of the numbers 1 through 10 is chosen at random. You are to try and guess the number chosen by asking questions with "yes-no" answers. Compute the expected number of questions that you need to ask in each of the following two cases:*

(a) *Your ith question is to be, "Is it i?", for $i = 1, \ldots, 10$.*

Solution. Let Q denote the number of questions asked. It follows that $P\{Q = i\} = 1/10$. Therefore,

$$E(Q) = \frac{1}{10} + \frac{2}{10} + \cdots + \frac{10}{10} = 5.5$$

(b) *With each question you try to eliminate one-half of the remaining numbers, as nearly as possible.*

Solution. Consider the following strategy: Divide what you have in halves and ask, "is it greater than or equal to x?" where x is the middle of the numbers. For instance, your first question is, "is it greater than or equal to 5.5?". [All other strategies of this type are similar. But you need to be consistent.] Let X denote the random number, so that $P\{X = i\} = 1/10$ for $i = 1, \ldots, 10$. If $X = 1, 2, 3, 6, 7, 8$, then $Q = 3$. If $X = 4, 5, 9, 10$ then $Q = 4$ (check!). Therefore, $P\{Q = 3\} = P\{X = 1\} + P\{X = 2\} + P\{X = 3\} + P\{X = 6\} + P\{X = 7\} + P\{X = 8\} = 0.6$, and $P\{Q = 4\} = 0.4$. Thus, $E(Q) = (3 \times 0.6) + (4 \times 0.4) = 3.4$.

3. *A person tosses a fair coin until a tail appears for the first time. If "tail" appears on the nth flip, then the person wins 2^n dollars. Let X denote the person's winnings. Show that $EX = \infty$. This is known as the "St.-Petersbourg paradox."*

Solution. For all $n \geq 1$,

$$P\{X = n\} = P(H_1 \cap \cdots \cap H_{n-1} \cap T_n) = \frac{1}{2^n}.$$

Else, $P\{X = n\} = 0$. Therefore,

$$EX = \sum_{n=1}^{\infty} \left(2^n \times \frac{1}{2^n}\right) = \infty.$$

4. *A box contains 5 red and 5 blue marbles. Two marbles are withdrawn randomly. If they are the same color, then you win \$1.10; if they are different colors, then you win −\$1.00 (i.e., you lose one dollar). Compute:*

(a) *the expected value of the amount you win;*

Solution. Let R_i denote the event, "red on ith draw." Then, $P(R_1 \cap R_2) = (5/10)(4/9)$ and $P(R_1^c \cap R_2^c) = (5/10)(4/9)$. Let X denote the win, where $X < 0$ means loss. Then, $P\{X = 1.10\} = 2 \times (5/10)(4/9) = 4/9$, and $P\{X = -1\} = 5/9$. Therefore, $EX = (1.1 \times \frac{4}{9}) - \frac{5}{9} = -0.0\bar{6}$.

(b) *the variance of the amount you win.*

Solution. Evidently, $E[X^2] = (1.1 \times \frac{4}{9}) + \frac{5}{9} = 1.0\bar{4}$. Therefore, $\text{Var}(X) = 1.0\bar{4} - (-0.0\bar{6})^2 = 1.04$.

Theoretical Problems:

1. *Let X be such that $P\{X = 1\} = p$ and $P\{X = -1\} = 1 - p$. Find a constant $c \neq 1$ such that $E[c^X] = 1$.*

Solution. First, for all real c,

$$E[c^X] = cp + c^{-1}(1-p).$$

Set this equal to one to find that $cp + (1-p)/c = 1$. I.e., $pc^2 - c + (1-p) = 0$. This is a quadratic equation (in c) and has two roots:

$$c = \frac{1 \pm \sqrt{1 - 4p(1-p)}}{2p}.$$

But $1 - 4p(1-p) = 1 - 4p + 4p^2 = (1-2p)^2$. Therefore,

$$c = \frac{1 \pm (1-2p)}{2p} = 1 \text{ and } \frac{1-p}{p}.$$

So the answer is $c = (1-p)/p$.

2. *Prove that if X is a Poisson random variable with parameter λ, then for all $n \geq 1$,*

$$E[X^n] = \lambda E[(X+1)^{n-1}]. \tag{1}$$

Use this to compute EX, $E[X^2]$, $\text{Var}X$, and $E[X^3]$.

Solution. We calculate directly:

$$\begin{aligned} E[X^n] &= \sum_{k=0}^{\infty} k^n P\{X = k\} \\ &= \sum_{k=0}^{\infty} k^n \frac{e^{-\lambda}\lambda^k}{k!} = \sum_{k=1}^{\infty} k^n \frac{e^{-\lambda}\lambda^k}{k!} \\ &= \lambda \sum_{k=1}^{\infty} k^{n-1} \frac{e^{-\lambda}\lambda^{k-1}}{(k-1)!} = \lambda \sum_{j=0}^{\infty} (j+1)^{n-1} \frac{e^{-\lambda}\lambda^j}{j!} = \lambda E[(X+1)^{n-1}]. \end{aligned}$$

Therefore,

$$\begin{aligned} EX &= \lambda E[(1+X)^0] = \lambda, \\ E[X^2] &= \lambda E[(1+X)] = \lambda(1+\lambda) = \lambda + \lambda^2, \\ \text{Var}X &= E[X^2] - (EX)^2 = \lambda, \\ E[X^3] &= \lambda E[(1+X)^2] = \lambda \left\{ E\left[1 + 2X + X^2\right] \right\} \\ &= \lambda \left\{ 1 + 2\lambda + \lambda + \lambda^2 \right\} = \lambda + 3\lambda^2 + \lambda^3. \end{aligned}$$