Solutions to Assignment #4
Math 501–1, Spring 2006
University of Utah

Problems:

1. *Suppose the distribution function of X is given by*

$$F(x) = \begin{cases} 0, & \text{if } x < 0, \\ 1/2, & \text{if } 0 \leq x < 1, \\ 3/5, & \text{if } 1 \leq x < 2, \\ 4/5, & \text{if } 2 \leq x < 3, \\ 9/10, & \text{if } 3 \leq x < 3.5, \\ 1, & \text{if } x \geq 3.5. \end{cases}$$

Compute the mass function of X.

Solution: Note that $P\{X \leq x\} = P\{X < x\} + P\{X = x\}$. That is, $F(x) = F(x-) + p(x)$. Solve to find that $p(x) = F(x) - F(x-)$ for all x. That is, $p(x)$ is the size of the jump of F at x (if any). Therefore,

$$p(x) = \begin{cases} 1/2, & \text{if } x = 0, \\ (3/5) - (1/2) = 1/10, & \text{if } x = 1, \\ (4/5) - (3/5) = 1/5, & \text{if } x = 2, \\ (9/10) - (4/5) = 1/10, & \text{if } x = 3, \\ 1 - (9/10) = 1/10, & \text{if } x = 3.5, \\ 0, & \text{otherwise.} \end{cases}$$

2. *Five distinct numbers are randomly distributed to players numbered 1 through 5. Whenever two players compare their numbers, the one with the higher one is declared the winner. Initially, players 1 and 2 compare their numbers; the winner then compares with player 3; and so on. Let X denote the number of times player 1 is declared a winner. Find the mass function of X. Use this to find the probability that X is even.*

Solution: Let Y_j denote the number distributed to player j. Note that $(Y_1, \ldots, Y_5)$ is a random permutation of $(1, \ldots, 5)$, all permutations being equally likely. Therefore, $p(0) = P\{X = 0\} = P\{Y_1 < Y_2\}$. But half of all permutations of $(1, \ldots, 5)$ have $Y_1 < Y_2$, whereas half have $Y_1 > Y_2$. Therefore,

$$p(0) = \frac{\frac{1}{2} \times 5!}{5!} = \frac{1}{2}.$$

Next note that $p(1) = P\{Y_2 < Y_1 < Y_3\}$. The number of ways to end up with $Y_2 < Y_1 < Y_3$ is the same as the number of ways to get $Y_1 < Y_2 < Y_3$. This is the same as .... Therefore, the number of ways to get $Y_2 < Y_1 < Y_3$ is $1/3!$ times the total number of permutations. That is,

$$p(1) = \frac{\frac{1}{3!} \times 5!}{5!} = \frac{1}{6}.$$

Next note that

$$\begin{aligned}p(2) &= P\{Y_2 < Y_1\,, Y_3 < Y_1\,, Y_4 > Y_1\}\\ &= P\{Y_1 = 3\,, Y_2 = 1\,, Y_3 = 2\} + P\{Y_1 = 3\,, Y_2 = 2\,, Y_3 = 1\}\\ &\qquad + P\{Y_1 = 4\,, Y_2 = 1\,, Y_3 = 2\,, Y_4 = 5\} + P\{Y_1 = 4\,, Y_2 = 2\,, Y_3 = 1\,, Y_4 = 5\}\\ &\qquad + P\{Y_1 = 4\,, Y_2 = 1\,, Y_3 = 3\,, Y_4 = 5\} + \cdots\\ &= \left(\frac{2}{5!} + \frac{2}{5!}\right) + \left(\frac{1}{5!} + \frac{1}{5!} + \cdots\right)\\ &= \left(2 \times \frac{2}{5!}\right) + \left(6 \times \frac{1}{5!}\right) = \frac{1}{12}.\end{aligned}$$

Next we note that

$$\begin{aligned}p(3) &= P\{Y_2 < Y_1\,, Y_3 < Y_1\,, Y_4 < Y_1\,, Y_5 > Y_1\}\\ &= P\{Y_1 = 4\,, Y_1 = 1\,, Y_2 = 2\,, Y_3 = 3\,, Y_4 = 4\,, Y_5 = 5\} + \cdots\\ &= \frac{4!}{5!} = \frac{1}{5}.\end{aligned}$$

Finally,

$$p(4) = P\{Y_1 = 5\} = \frac{1}{5}.$$

Consequently, $P\{X \text{ is even}\} = p(2) + p(4) = (1/12) + (1/5) = 17/60 = 0.283\bar{3}$. [Some people think of 0 as even. Then for them this probability is 47/60. Both answers are correct, depending on your interpretation.]

3. *Four independent flips of a fair coin are made. Let X denote the number of heads so obtained. Find the mass function of $X - 2$, and plot this function.*

Solution: Let $Z := X - 2$. Then, the possible values of Z are $0, \pm 1, \pm 2$. Note that $P\{Z = 0\} = P\{X = 2\} = \binom{4}{2}(1/2)^4 = 3/8$, $P\{Z = 1\} = P\{X = 3\} = \binom{4}{3}(1/2)^4 = 1/4$, $P\{Z = -1\} = P\{X = 1\} = \binom{4}{1}(1/2)^4 = 1/4$, $P\{Z = 2\} = P\{X = 4\} = \binom{4}{4}(1/2)^4 = 1/16$, and $P\{Z = -2\} = P\{X = 0\} = \binom{4}{0}(1/2)^4 = 1/16$. Plotting is left up to you.

4. *A fair coin is continually flipped until heads appear for the tenth time. Let X denote the number of tails that occur in the mean time. Compute $P\{2 \le X \le 5\}$.*

Solution: Note that $X = k$ if and only if the tenth head occurs on trial $10 + k$. Therefore, $P\{X = k\}$ can be found from hypergeometric probabilities:

$$P\{X = k\} = \binom{9+k}{k}\frac{1}{2^{10+k}}, \qquad k = 0, 1, 2, \ldots.$$

Theoretical Problems:

1. *Let X be a negative binomial random variable with parameters r and p, and let Y be a binomial random variable with parameters n and p. Prove that*

$$P\{X > n\} = P\{Y < r\}.$$

Solution: Perform independent Bernoulli trials, where the success-per-trial probability is p. Then X is the first time we get the rth success, so that $P\{X > n\}$ is the probability that the total number of successes in the first n trials is strictly less than r. Because Y *is* the total number of successes in the first n trials, we have the identity.

2. *Suppose X is geometric with parameter p. That is, $P\{X = k\} = p(1-p)^{k-1}$ for $k = 1, 2, \ldots$. Then:*

(a) *Compute explicitly $P\{X \geq n\}$ for any positive integer n.*

Solution: This is just summing a geometric series; i.e.,

$$\begin{aligned}
P\{X \geq n\} &= \sum_{j=n}^{\infty} P\{X = j\} \\
&= p\sum_{j=n}^{\infty}(1-p)^{j-1} = p\sum_{\ell=n-1}^{\infty}(1-p)^{\ell} \\
&= p \cdot \frac{(1-p)^n}{1-(1-p)} = (1-p)^n.
\end{aligned}$$

(b) *Compute explicitly $P\{X = n+k \,|\, X \geq n\}$. Compare your answer to $P\{X = k\}$.*

Solution: The conditional probability that we seek is

$$\begin{aligned}
P\{X = n+k \,|\, X \geq n\} &= \frac{P\{X = n+k\,, X > n\}}{P\{X \geq n\}} = \frac{P\{X = n+k\}}{P\{X \geq n\}} \\
&= \frac{p(1-p)^{n+k-1}}{(1-p)^n} = p(1-p)^{k-1} \\
&= P\{X = k\}.
\end{aligned}$$

The original formulation of the question asked for $P\{X = n+k \,|\, X > n\}$. A similar calculation shows that this is equal to $p(1-p)^{k-2} = P\{X = k\}/(1-p)$.