Solutions to Problem Assignment #1
Math 501–1, Spring 2006
University of Utah

Problems:

1. *Twenty workers are to be assigned to* 20 *different jobs, one to each job. How many different assignments are possible?*

Solution: $20! \approx 2.43 \times 10^{18}$.

2. *Consider a group of* 20 *people. If everyone shakes hands with everyone else, then how many handshakes take place?*

Solution: $\binom{20}{2} = \frac{20!}{2! \times 18!} = \frac{20 \times 19}{2!} = 190$.

3. *Five separate awards (best scholarship, best leadership qualities, and so on) are to be presented to selected students from a class of* 30*. How many different outcomes are possible if:*

(a) *a student can receive any number of awards;*

Solution: $30^5 = 24,300,000$.

(b) *each student can receive at most* 1 *award?*

Solution: $30 \times 29 \times 28 \times 27 \times 26 = 17,100,720$.

4. *A person has* 8 *friends, of whom* 5 *will be invited to a party.*

(a) *How many choices are there if* 2 *of the friends are feuding and will not attend together?*

Solution: There are a total of $\binom{8}{5} = \frac{8!}{3! \times 5!} = 56$ ways to form invitations. But many of them contain the feuding duo. The number of possible invitations that contain the feuding duo is, in fact, $\binom{6}{3} = 20$. Therefore, there are $56 - 20 = 36$ possible invitations that do not include both of the fighting pair.

(b) *How many choices if* 2 *of the friends will only attend together?*

Solution: There are 20 possible ways for inviting the two. Also, there are $\binom{6}{5} = 6$ ways of not inviting them. Thus, there are 26 many possible invitations of this type.

Theoretical Problems:

1. *Verify that* $\binom{n}{k} = \binom{n}{n-k}$*. Use this to prove that*

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2.$$

It is easy to see that $\binom{n}{k} = \frac{n!}{k! \times (n-k)!} = \frac{n!}{(n-k)! \times k!} = \binom{n}{n-k}$. Now, $\binom{2n}{n}$ is the number of ways of forming a team of n people from $2n$. Now concentrate on the $2n$ people. Our team could be formed by either choosing:

#1. 0 people from the first n and n people from the second n; or

#2. 1 person from the first n and $n-1$ people from the second n; or $\cdots$

$\vdots$

#n. n people from the first n and 0 people from the second n.

Items #1 through #n cannot be done simultaneously. So they represent different ways in total. For #k, the number of choices are $\binom{n}{k}\binom{n}{n-k} = \binom{n}{k}^2$. Therefore, there are $\sum_{k=0}^{n} \binom{n}{k}^2$-many ways of creating our team. But this must be equal to $\binom{2n}{n}$.