Exam 5, Math 5010-1

Fall 2016
November 30, 2016

This is a 50-minute exam. You may use your textbook, as well as a calculator, but your work must be completely yours.

The exam is made of 3 questions in 3 pages, and is worth 35 points, total. Be sure to try all of the problems.

Partial cblueit is given only to carefully-written solutions.

1. (15 points total) Let X and Y be two independent random variables, both distributed uniformly in $[0\,,1]$. Compute the following quantities:

   (a) (5 points) $\mathrm{E}(XY)$.

   Solution. By independence,

   $$\mathrm{E}(XY) = \mathrm{E}(X)\mathrm{E}(Y) = \frac{1}{4}.$$

   (b) (5 points) $\mathrm{E}(X - Y)$.

   Solution. By the linearity of expectations,

   $$\mathrm{E}(X - Y) = \mathrm{E}(X) - \mathrm{E}(Y) = \frac{1}{2} - \frac{1}{2} = 0.$$

   (c) (5 points) $\mathrm{E}\left([X - Y]^2\right)$.

   Solution. Here we have to do some real work:

   $$\begin{aligned}\mathrm{E}\left([X - Y]^2\right) &= \int_0^1 \int_0^1 (x-y)^2 f(x\,,y)\,dx\,dy = \int_0^1 \int_0^1 (x-y)^2\,dx\,dy \\ &= \int_0^1 \left(\int_0^1 \left[x^2 - 2xy + y^2\right] dx \right) dy = \int_0^1 \left(\frac{1}{3} - y + y^2 \right) dy \\ &= \frac{1}{3} - \frac{1}{2} + \frac{1}{3} = \frac{1}{6}.\end{aligned}$$

2. (15 points total) Suppose X is distributed uniformly on $[0\,,2\pi]$. Define

$$U := \sin(X), \qquad V := \cos(X).$$

(a) (5 points) Are U and V independent? Prove or disprove.

Solution. No. In fact, $U^2 + V^2 = 1$.

(b) (5 points) Compute $\mathrm{Cov}(U\,,V)$. You may use the following trigonometric identity without proof: $\sin(\theta)\cos(\theta) = \frac{1}{2}\sin(2\theta)$.

Solution. Use the law of the unconscious statistician:

$$\mathrm{E}(UV) = \int_0^{2\pi} \sin(x)\cos(x)\frac{dx}{2\pi} = \frac{1}{2\pi}\int_0^{2\pi} \sin(2x)\,dx = 0,$$

$$\mathrm{E}(U) = \int_0^{2\pi} \sin(x)\frac{dx}{2\pi} = 0,$$

and

$$\mathrm{E}(V) = \int_0^{2\pi} \cos(x)\frac{dx}{2\pi} = 0.$$

Therefore, $\mathrm{Cov}(U\,,V) = \mathrm{E}(UV) - \mathrm{E}(U)\mathrm{E}(V) = 0$.

(c) (5 points) Are U are V uncorrelated? Prove or disprove.

Solution. Yes. $\mathrm{Cov}(U\,,V) = 0$, and the correlation is covariance divided by the product of the standard deviations. Therefore, the correlation is zero.

3. (5 points) Suppose X and Y are two random variables with

$$\mathrm{E}(X) = 1,\ \mathrm{E}(Y) = 2,\ \mathrm{E}(XY) = 3,\ \mathrm{Var}(X) = 9,\ \mathrm{Var}(Y) = 4.$$

What is the numerical value of $\mathrm{Var}(X - 2Y)$? Show your work.

Solution.

$$\begin{aligned}
\mathrm{Var}(X - 2Y) &= \mathrm{Var}(X) + \mathrm{Var}(-2Y) + 2\mathrm{Cov}(X\,, -2Y) \\
&= \mathrm{Var}(X) + 4\mathrm{Var}(Y) - 4\mathrm{Cov}(X\,, Y) \\
&= 9 + (4 \times 4) - 4\,(\mathrm{E}[XY] - \mathrm{E}[X]\mathrm{E}[Y]) \\
&= 9 + 16 - 4\,(3 - 2) = 21.
\end{aligned}$$