Exam 4, Math 5010-1

Fall 2016
November 9, 2016

This is a 50-minute exam. You may use your textbook, as well as a calculator, but your work must be completely yours. The exam is made of 5 questions in 5 pages, and is worth 45 points, total. Be sure to try all of the problems. Partial credit is given only to carefully-written solutions.

1. (10 points each) Suppose X takes on the values 0, ± 1, ± 2, or ± 3 with equal probability. Let $Y = X^2$.

   (a) (5 points) What is the pmf of Y?
   Solution. For every real number y,

   $$f_Y(y) = \mathrm{P}\{Y = y\} = \mathrm{P}\{X^2 = y\} = \mathrm{P}\{X = y\} + \mathrm{P}\{X = -y\}.$$

   Therefore,

   $$f_Y(y) = \begin{cases} 1/7 & \text{if } y = 0, \\ 2/7 & \text{if } y = 1 \text{ or } y = 4 \text{ or } y = 9, \\ 0 & \text{otherwise.} \end{cases}$$

   (b) (5 point) Use your answer to part (a) to compute $\mathrm{P}\{Y \leq 3\}$.
   Solution. $\mathrm{P}\{Y \leq 3\} = f_Y(0) + f_Y(1) = 3/7$.

2. (10 points total) Let $Y = X^2$ where X is distributed uniformly on the interval $[-1\,,1]$.

(a) (5 points) Compute the probability density function of Y.

Solution. $F_Y(y) = \mathrm{P}\{Y \le y\} = \mathrm{P}\{X^2 \le y\}$ for all real numbers y. Therefore, $F_Y(y) = 0$ if $y \le 0$ and $F_Y(y) = 1$ if $y \ge 1$; and for $0 < y < 1$,

$$\begin{aligned} F_Y(y) &= \mathrm{P}\{|X| \le \sqrt{y}\} = \mathrm{P}\{-\sqrt{y} \le X \le \sqrt{y}\} \\ &= \int_{-\sqrt{y}}^{\sqrt{y}} f_X(x)\,\mathrm{d}x = \int_{-\sqrt{y}}^{\sqrt{y}} \tfrac{1}{2}\,\mathrm{d}x = \sqrt{y}. \end{aligned}$$

Therefore,

$$f_Y(y) = F_Y'(y) = \begin{cases} \dfrac{1}{2\sqrt{y}} & \text{if } 0 < y < 1, \\ 0 & \text{otherwise.} \end{cases}$$

(b) (5 points) Use your answer to part (a) to compute $\mathrm{P}\{Y \le 1/2\}$.

Solution.

$$\mathrm{P}\{Y \le 1/2\} = \int_0^{1/2} \frac{1}{2\sqrt{y}}\,\mathrm{d}y = \sqrt{2y}\,\Big]_0^{1/2} = \frac{1}{\sqrt{2}}.$$

This ought to make sense because we can also compute $\mathrm{P}\{Y \le 1/2\}$ as follows:

$$\mathrm{P}\{Y \le 1/2\} = \mathrm{P}\left\{|X| \le \frac{1}{\sqrt{2}}\right\} = \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \tfrac{1}{2}\,\mathrm{d}x = \frac{1}{\sqrt{2}}.$$

3. (10 points total) Suppose X and Y are independent, both distributed uniformly on $[0\,,1]$.

(a) (5 points) Compute the cdf of $X+Y$.

Solution. We know that $f_X(a) = f_Y(a) = 1$ if $0 < a < 1$ and $f_X(a) = f_Y(a) = 0$ otherwise. By independence, $f(x\,,y) = f_X(x)f_Y(y)$, and hence

$$F_{X+Y}(a) = \mathrm{P}\{X+Y \le a\} = \iint_{x+y\le a} f(x\,,y)\,\mathrm{d}x\,\mathrm{d}y = \iint_{\substack{0<x,y<1\\ x+y\le a}} \mathrm{d}x\,\mathrm{d}y.$$

You need to draw the region in order to see that there are two cases: Case 1. $0 < a < 1$; Case 2. $1 \le a < 2$. In all other cases, $F_{X+Y}(a) = 0$.

When $0 < a < 1$:

$$F_{X+Y}(a) = \int_0^a \mathrm{d}y \int_0^{a-y} \mathrm{d}x = \int_0^a (a-y)\,\mathrm{d}y = \frac{a^2}{2}.$$

When $1 \le a < 2$:

$$F_{X+Y}(a) = 1 - \mathrm{P}\{X+Y > a\} = 1 - \int_{a-1}^1 \mathrm{d}y \int_{a-y}^1 \mathrm{d}x = 1 - \int_{a-1}^1 (1-a+y)\,\mathrm{d}y$$
$$= 1 - \int_0^{2-a} z\,\mathrm{d}z = 1 - \frac{(2-a)^2}{2}.$$

(b) (5 points) Use your answer to part (a) in order to find the pdf of $X+Y$.

Solution. Differentiate F_{X+Y}:

$$f_{X+Y}(a) = F'_{X+Y}(a) = \begin{cases} a & \text{if } 0 < a < 1, \\ 2-a & \text{if } 1 < a < 2. \end{cases}$$

4. (10 points total) Suppose X has a $\mathrm{Bin}(n\,, p)$ distribution, where $n \geq 3$. It might help to recall that $\mathrm{E}(X) = np$ and $\mathrm{E}(X^2) = np - np^2 + n^2p^2$.

   (a) (5 points) Compute $\mathrm{E}[X(X-1)(X-2)]$.
   Solution.

$$
\begin{aligned}
\mathrm{E}[X(X-1)(X-2)] &= \sum_{k=0}^{n} k(k-1)(k-2)\binom{n}{k}p^k(1-p)^{n-k} \\
&= \sum_{k=3}^{n} k(k-1)(k-2)\binom{n}{k}p^k(1-p)^{n-k} \\
&= \sum_{k=3}^{n} \frac{n!}{(n-k)!(k-3)!}p^k(1-p)^{n-k} \\
&= n \times (n-1) \times (n-2)p^3 \sum_{k=3}^{n}\binom{n-3}{k-3}p^{k-3}(1-p)^{[n-3]-[k-3]} \\
&= n(n-1)(n-2)p^3.
\end{aligned}
$$

   (b) (5 points) Use your answer to part (a) to compute $\mathrm{E}(X^3)$.
   Solution. Since $X(X-1)(X-2) = (X^2 - X)(X-2) = X^3 - X^2$,

$$
\begin{aligned}
n(n-1)(n-2)p^3 &= \mathrm{E}[X(X-1)(X-2)] = \mathrm{E}(X^3) - \mathrm{E}(X^2) \\
&= \mathrm{E}(X^3) - np(1-p+np).
\end{aligned}
$$

   Solve to obtain

$$
\mathrm{E}(X^3) = n(n-1)(n-2)p^3 + np(1-p+np).
$$

5. (5 points) A man and a woman agree to meet at a certain location about 12:30 p.m. If the man arrives at a time uniformly distributed between 12:15 and 12:45, and if the woman independently arrives at a time uniformly distributed between 12:00 and 1:00 p.m., then find the probability that the first to arrive waits no longer than 5 minutes.

Solution. Let X and Y respectively denote the waiting times of the man and the woman, measured in minutes past noon. We know that:

(a) X and Y are independent;

(b) $f_X(x) = 1/30$ if $15 \le x \le 45$ and $f_X(x) = 0$ otherwise; and

(c) $f_Y(y) = 1/60$ if $0 \le y \le 60$ and $f_Y(y) = 0$ otherwise.

Therefore,

$$f(x\,, y) = \frac{1}{1800} \qquad \text{for } (x\,, y) \text{ in the green region below,}$$

and $f(x\,, y) = 0$ otherwise. We are asked to find $\mathrm{P}\{|X - Y| \le 5\}$. Let $\mathcal{R}$ denote the intersection of the green and orange regions in the figure below. Then,

$$\mathrm{P}\{|X - Y| \le 5\} = \iint_{\mathcal{R}} \frac{1}{1800}\, dx\, dy = \int_{15}^{45} \left(\int_{x-5}^{x+5} \frac{1}{1800}\, dy \right) dx = \frac{1}{6}.$$

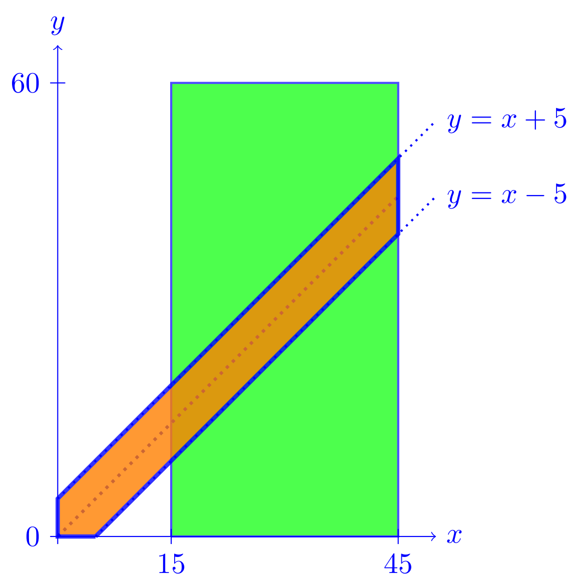

Figure 1: The green region is where the pdf is non zero. The region of interest is the intersection of the green and orange regions.