Exam 3, Math 5010-1

Fall 2016
October 19, 2016

This is a 50-minute exam. You may use your textbook, as well as a calculator, but your work must be completely yours.

The exam is made of 5 questions in 5 pages, and is worth 35 points, total. Be sure to try all of the problems.

Partial credit is given only to carefully-written solutions.

1. (5 points) Suppose a random variable X has an exponential distribution with parameter $\lambda > 0$, and define $Y = \ln X$. What is the probability density function of Y?

   Solution. We know that

   $$f_X(x) = \lambda e^{-\lambda x} \text{ if } x > 0, \text{ and } f_X(x) = 0 \text{ otherwise.}$$

   Now,

   $$F_Y(y) = P\{Y \leq y\} = P\{X \leq e^y\} = F_X(e^y).$$

   Differentiate to find that, because $e^y \geq 0$ for all y,

   $$f_Y(y) = f_X(e^y)e^y = \lambda \exp\{-\lambda e^y + y\} \qquad \text{for all } -\infty < y < \infty.$$

2. (5 points) Consider n independent binomial trials where the success probability is p per trial. Let X denote the total number of successes. Prove that, if $X = k$, then all $\binom{n}{k}$ possible arrangements of k successes and $n-k$ failures are equally likely.

Solution. Before we start to compute thing, we first have to parse the problem carefully: X is a random variable. Therefore, the statement "if $X = k$" means that $\boxed{\text{we are asked to compute a conditional probability given } \{X = k\}.}$ For instance, if $n = 2$ and $k = 1$, then we are asked to show that

$$\mathrm{P}(S_1 \cap S_2^c \mid X = 1) = \mathrm{P}(S_1^c \cap S_2 \mid X = 1),$$

where S_j refers to the event that the jth trial has led to a success.

Once we understand the question we can proceed as follows: We know that

$$\mathrm{P}\{X = k\} = \begin{cases} \dbinom{n}{k} p^k (1-p)^{n-k} & \text{for } k = 0, \ldots, n, \\ 0 & \text{for all other values of } k. \end{cases}$$

Let E denote the event that a given arrangement of spots leads to success, and the rest to failure. [For example, E could denote the event that exactly the first k spots are successes.] Then, $E \cap \{X = k\} = E$ by elementary logic, and hence

$$\mathrm{P}(E \mid X = k) = \frac{\mathrm{P}(E \cap \{X = k\})}{\mathrm{P}\{X = k\}} = \frac{\mathrm{P}(E)}{\mathrm{P}\{X = k\}} = \frac{p^k(1-p)^{n-k}}{\dbinom{n}{k} p^k (1-p)^{n-k}} = \frac{1}{\dbinom{n}{k}},$$

for any integer $k = 0, \ldots, n$. In other words, for any given arrangement of k successes and $n-k$ failures, the conditional probability, given that $X = k$, is $\binom{n}{k}^{-1}$. This of course does the job.

3. (10 points total) A certain random variable X has the following cumulative distribution function F,

$$F(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1/2 & \text{if } 0 \leq x < 1, \\ 3/4 & \text{if } 1 \leq x < 2, \\ 1 & \text{if } x \geq 2. \end{cases}$$

(a) (4 points) Compute $\mathrm{P}\{X < 1.5\}$.

Solution. $\mathrm{P}\{X < 1.5\} = F(1.5-) = F(1.5) = 3/4$.

(b) (6 points) Is X discrete or continuous? If it is discrete, then what is its probability mass function? If X is continuous, then what is its probability density function?

Solution. X is discrete with mass function

$$f(x) = \begin{cases} 1/2 & \text{if } x = 0, \\ 1/4 & \text{if } x = 1, \\ 1/4 & \text{if } x = 2. \end{cases}$$

4. (5 points) Let X be a random variable with a Poisson distribution with parameter $\lambda > 0$. Show that

$$\mathrm{P}\{X \text{ is odd}\} = \frac{1 - \mathrm{e}^{-2\lambda}}{2} = \frac{\sinh(\lambda)}{\mathrm{e}^{\lambda}}.$$

(Hint. Begin by Taylor expanding $g(x) = \sinh(x) = \frac{1}{2}(\mathrm{e}^x - \mathrm{e}^{-x})$ at $x = 0$.)

Solution. We follow the hint and Taylor expand $g = \sinh$. First,

$$g'(x) = \frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2}, \quad g''(x) = \frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2}, \ldots .$$

Therefore,

$$\begin{aligned} g(x) &= g(0) + g'(0)x + g''(0)\frac{x^2}{2} + g'''(0)\frac{x^3}{3!} + \cdots \\ &= 0 + x + 0 + \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + \cdots \\ &= \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}. \end{aligned}$$

Now,

$$\mathrm{P}\{X \text{ is odd}\} = \sum_{k=0}^{\infty} \frac{\mathrm{e}^{-\lambda}\lambda^{2k+1}}{(2k+1)!} = \mathrm{e}^{-\lambda} g(\lambda) = \frac{\sinh(\lambda)}{\mathrm{e}^{\lambda}}.$$

5. (10 points total) Let X be a random variable with probability density function

$$f(x) = \tfrac{1}{2}\mathrm{e}^{-|x|} \quad \text{for all } -\infty < x < \infty.$$

(a) (5 points) Compute $\mathrm{P}\{X \le -1\}$.
Solution. If $a \le 0$, then

$$F(a) = \int_{-\infty}^{a} \tfrac{1}{2}\mathrm{e}^{-|x|}\,\mathrm{d}x = \tfrac{1}{2}\int_{-\infty}^{a} \mathrm{e}^{x}\,\mathrm{d}x = \frac{\mathrm{e}^{a}}{2}.$$

Therefore, $\mathrm{P}\{X \le -1\} = F(-1) = 1/(2\mathrm{e})$.

(b) (5 points) Compute $\mathrm{P}\{X \le 2\}$.
Solution. If $a > 0$, then

$$F(a) = \int_{-\infty}^{a} \tfrac{1}{2}\mathrm{e}^{-|x|}\,\mathrm{d}x = F(0) + \int_{0}^{a} \tfrac{1}{2}\mathrm{e}^{-x}\,\mathrm{d}x = \frac{1}{2} + \frac{1-\mathrm{e}^{-a}}{2} = 1 - \frac{\mathrm{e}^{-a}}{2}.$$

Therefore, $\mathrm{P}\{X \le 2\} = F(2) = 1 - \tfrac{1}{2}\mathrm{e}^{-2}$.