Exam 1, Math 5010-1

Fall 2016
September 7, 2016

> This is a 50-minute exam. You may use your textbook, as well as a calculator, but your work must be completely yours.
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> The exam is made of 4 questions in 4 pages, and is worth 30 points, total. Be sure to try all of the problems.
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> Partial credit is given only to carefully-written solutions.

1. (8 points total; 2 points each) Simplify the following sets; justify your answers:

   (a) $\cup_{n=1}^{\infty}(1-\frac{1}{n}\,,1+\frac{1}{n})$.
   Solution. $x \in \cup_{n=1}^{\infty}(1-\frac{1}{n}\,,1+\frac{1}{n})$ if and only if $x \in (1-\frac{1}{n}\,,1+\frac{1}{n})$ for some $n \geq 1$. This implies that $\cup_{n=1}^{\infty}(1-\frac{1}{n}\,,1+\frac{1}{n}) = (0\,,2)$.

   (b) $\cap_{n=1}^{\infty}(1-\frac{1}{n}\,,1+\frac{1}{n})$.
   Solution. $x \in \cap_{n=1}^{\infty}(1-\frac{1}{n}\,,1+\frac{1}{n})$ if and only if $x \in (1-\frac{1}{n}\,,1+\frac{1}{n})$ for all $n \geq 1$. This implies that $\cap_{n=1}^{\infty}(1-\frac{1}{n}\,,1+\frac{1}{n}) = \{1\}$.

   (c) $\cap_{n=1}^{\infty}(1-\frac{1}{n}\,,1)$.
   Solution. $x \in \cap_{n=1}^{\infty}(1-\frac{1}{n}\,,1)$ if and only if $x \in (1-\frac{1}{n}\,,1)$ for every $n \geq 1$. There is no such point x; therefore, $\cap_{n=1}^{\infty}(1-\frac{1}{n}\,,1) = \varnothing$.

   (d) $\cap_{n=1}^{\infty}(1-\frac{1}{n}\,,1]$.
   Solution. $x \in \cap_{n=1}^{\infty}(1-\frac{1}{n}\,,1]$ if and only if $x \in (1-\frac{1}{n}\,,1]$ for every $n \geq 1$. The only such point is $x = 1$. Therefore, $\cap_{n=1}^{\infty}(1-\frac{1}{n}\,,1] = \{1\}$.

2. (4 points) A police department in a small city consists of 15 officers. The department policy is to have 10 of the officers patrolling the streets, 2 of the officers working full time at the station, and 3 of the officers on reserve at the station. How many different divisions of the 15 officers into the 3 groups are possible?

Solution. Since this is a question of unordered arrangements, there are

$$\frac{15!}{10! \cdot 2! \cdot 3!}$$

possible arrangements.

3. (9 points total; 3 points each) We select four letters at random from the english alphabet (26 letters, a–z) in order to create a code. Repeats are possible.

   (a) Identify your sample space Ω.

   Solution. Ω is the collection of all ordered 4-letter codes in English.

   (b) How many elements does Ω have?

   Solution. Each of our four letters has 26 possible outcomes; therefore, there are 26^4 possible such codes.

   (c) If all possible four-letter outcomes are equally likely, then compute the probability that our code begins with a vowel?

   Solution. Let A denote the collection of all possible 4-letter codes that begin with a vowel. Since $|A| = 5 \times 26^3$,

$$\mathrm{P}(A) = \frac{5 \times 26^3}{26^4} = \frac{5}{26} \approx 0.1923.$$

4. (9 points; 3 points each) There are 500 people in a freshman class, half of whom are women. We select 5 freshmen at random to form a student council.

(a) What is the sample space Ω?

Solution. Ω denotes the collection of all possible unordered student councils of 5 students.

(b) How many elements does Ω have?

Solution. Since the selection is unordered,

$$|\Omega| = \binom{500}{5}.$$

(c) If all possible student councils are equally likely, then what is the probability that the randomly-selected student council is made up of all women?

Solution. Let A denote the collection of all possible ways to select 5 women from the student body. Then,

$$|A| = \binom{250}{5},$$

and hence

$$\mathrm{P}\{\text{all women in the student council}\} = \frac{\binom{250}{5}}{\binom{500}{5}}.$$