Solutions to Midterm, Math 4200, Summer 2009

1. *Compute $\int_\gamma (1/z)\,\mathrm{d}z$ where γ is the straight line that starts at the point i and ends at 1.*

   Solution: We can parametrize the line with, say, the following path:

   $$\gamma(t) := t + (1-t)i = i + (1-i)t \quad \text{for } 0 \le t \le 1.$$

   Therefore,

   $$\int_\gamma \frac{\mathrm{d}z}{z} = \int_0^1 \frac{\gamma'(t)}{\gamma(t)}\,\mathrm{d}t = \int_0^1 (\log\gamma(t))'\,\mathrm{d}t = \log\frac{\gamma(1)}{\gamma(0)} = -\log i.$$

   Because $|i| = 1$ and $\arg(i) = \pi/2$ [using the principle branch of the argument], $\log i = \log|i| + i\arg(i) = \pi i/2$. Therefore,

   $$\int_\gamma \frac{\mathrm{d}z}{z} = -\frac{\pi i}{2}.$$

2. *Compute $\int_\gamma \mathrm{e}^z\,\mathrm{d}z$, where γ is a circle of radius 2 that goes around the point 1 two times (counterclockwise).*

   Solution 1: The function $f(z) = \mathrm{e}^z$ is analytic on all of $\mathbf{C}$; therefore, it is analytic on any convex open set U that contains the circle described here [e.g., U the circle of radius 5 around 1]. Cauchy's theorem for convex sets tells us that $\int_\gamma \mathrm{e}^z\,\mathrm{d}z = 0$.

   Solution 2: First we parametrize the circle:

   $$\gamma(t) = 1 + 2\mathrm{e}^{it} \quad \text{for } 0 \le t \le 4\pi.$$

Then compute directly, using the computation $\gamma'(t) = 2ie^{it}$:

$$\int_\gamma \mathrm{e}^z \,\mathrm{d}z = \int_0^{4\pi} \mathrm{e}^{\gamma(t)}\gamma'(t)\,\mathrm{d}t = \int_0^{4\pi} \left(\mathrm{e}^{\gamma(t)}\right)' \,\mathrm{d}t = \mathrm{e}^{\gamma(4\pi)} - \mathrm{e}^{\gamma(0)} = 0.$$

3. *State (without proof) the Cauchy–Riemann equations. Use them to prove that the real part* $u := \mathit{Re} f$ *of an analytic function is harmonic; i.e., satisfies* $u_{xx} + u_{yy} = 0$.

   Solution: The Cauchy–Riemann equations tell us that: (i) $u_x = v_y$; and (ii) $u_y = -v_x$. Now, $u_{xx} = (u_x)_x = (v_y)_x = (v_x)_y = (-u_y)_y = -u_{yy}$. Therefore, $u_{xx} + u_{yy} = 0$.

4. *Let* Δ *denote a triangle that goes around the origin. Prove that if* γ *traces the boundary of* Δ*—counterclockwise—three times, then* $\mathit{ind}_\gamma(0) = 3$*, no matter what* Δ *looks like.*

   Solution: Covered in lecture.