Solutions to Homework 6

Math 4200, Summer 2009

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#1, p. 124. No. Yes.

#1, p. 131. Suppose $\sin(\pi z) = 0$ for a complex number z. Because

$$\sin(\pi z) = \frac{e^{i\pi z} - e^{-i\pi z}}{2i},$$

it follows that $e^{i\pi z} = e^{-i\pi z}$, which is to say that $e^{2i\pi z} = 1$. That is, z is an integer.

#7, p. 132. This is really easy to do, thanks to the calculus of residues. I will leave that approach to you [excellent preparation]. Here is a direct method that is line with the material of the chapter.

Write

$$\frac{1}{z^2 - 1} = \frac{1}{(z-1)(z+1)} = \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right).$$

Then,

$$\int_\gamma \frac{f(z)}{z^2 - 1}\, dz = \frac{1}{2}\int_\gamma \frac{f(z)}{z-1}\, dz - \frac{1}{2}\int_\gamma \frac{f(z)}{z+1}\, dz.$$

Theorem 4.2.9 shows that

$$\int_\gamma \frac{f(z)}{z^2 - 1}\, dz = \frac{1}{2}\left(2\pi i f(1) - 2\pi i f(-1)\right) = 2\pi i \cdot \frac{f(1) - f(-1)}{2}.$$

#9, p. 132. We know from the homology version of Cauchy's theorem that

$$\mathrm{Ind}_\Gamma(z) f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(w)}{w - z}\, dw.$$

If $r < |z| < R$, then $\mathrm{Ind}_{\gamma_1}(z) = 0$ and $\mathrm{Ind}_{\gamma_2}(z) = 1$, and therefore $\mathrm{Ind}_\Gamma(z) = 1 - 0 = 1$. This has the desired result.

#4, p. 139. Because

$$e^{-z} = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!},$$

it follows that

$$\frac{e^{-z}}{z^3} = \sum_{k=0}^{\infty} \frac{(-z)^k}{k! z^3} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} z^{k-3} = \sum_{j=-3}^{\infty} \frac{(-1)^{j+3}}{(j+3)!} z^j = -\sum_{j=-3}^{\infty} \frac{(-1)^j}{(j+3)!} z^j.$$

That is, $c_j = 0$ if $j \leq -4$, and $c_j = (-1)^j/(j+3)!$ if $j \geq -3$.

#6, p. 139. Write

$$\frac{z}{z^2+1} = \frac{g(z)}{z-i},$$

where

$$g(z) = \frac{z(z-i)}{z^2+1} = \frac{z(z-i)}{(z-i)(z+i)} = \frac{z}{z+i}.$$

Note that

$$g'(z) = i(z+i)^{-2},\ g''(z) = -2i(z+i)^{-3}, \ldots,\ g^{(k)}(z) = (-1)^{k+1} k! i (z+i)^{-k-1}.$$

Therefore, we can power-expand g around $z = i$:

$$\begin{aligned} g(z) &= \sum_{k=0}^{\infty} \frac{(z-i)^k}{k!} g^{(k)}(i) \\ &= i \sum_{k=0}^{\infty} (-1)^{k+1} (z-i)^{-k-1}. \end{aligned}$$

And so,

$$\frac{z}{z^2+1} = i \sum_{k=0}^{\infty} (-1)^{k+1} (z-i)^{-k-2} = i \sum_{j=-2}^{\infty} (-1)^{j-1} (z-i)^j.$$

#2, p. 145. Let $f(z) = 1/(z^2 - 3z)$. We need to find the residues of f at $z = 0$ and $z = 3$. First, $\mathrm{Res}(f, 0)$: Write

$$f(z) = \frac{g(z)}{z}, \qquad \text{where } g(z) = \frac{1}{z-3}.$$

The function g is analytic near $z = 0$. Therefore, we can write it as $g(z) = g(0) + g'(0)z + \frac{1}{2} g''(0) z^2 + \cdots$ to find that the coefficient of z^{-1} in

f is $g(0) = -1/3$. That is, $\text{Res}(f\,, 0) = -1/3$.

Similarly we write

$$f(z) = \frac{g(z)}{z-3}, \qquad \text{where } g(z) = \frac{1}{z}.$$

And we are led to $\text{Res}(f\,, 3) = g(3) = 1/3$. Therefore,

$$\frac{1}{2\pi i}\int_\gamma f(z)\,dz = \text{Ind}_\gamma(0)\text{Res}(f\,,0) + \text{Ind}_\gamma(3)\text{Res}(f\,,3) = 0.$$

#4, p. 145. $f(z) := e^z - 1$ has a simple pole at $z = 0$. Therefore, write

$$f(z) = \frac{g(z)}{z}, \qquad \text{where } g(z) = \frac{z}{e^z - 1}.$$

As before, $\text{Res}(f\,, 0) = g(0) = \lim_{z\to 0} g(z)$, because the pole of f at zero is simple. Therefore,

$$\text{Res}(f\,,0) = \lim_{z\to 0}\frac{z}{z + \frac{z^2}{2} + \cdots} = \lim_{z\to 0}\frac{1}{1 + \frac{z}{2} + \cdots} = 1.$$

Consequently,

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{e^z - 1} = 1 \qquad \Rightarrow \qquad \int_\gamma \frac{dz}{e^z-1} = 2\pi i.$$