Solutions to Homework 5

Math 4200, Summer 2009

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#2, p. 101. Straight forward.

#3, p. 101. From #2 we know that $1/f(z) \to 0$ as $z \to \infty$. If f has no zeros in $\mathbf{C}$, then $1/f$ must be a bounded analytic function, and hence $1/f(z) = 0$ for all z by Liouville's theorem. But this is absurd because f is analytic. Therefore, f has zeros.

#5, p. 101. The function $g(z) := e^{f(z)}$ is analytic. Moreover, $|g(z)| = e^{\mathrm{Re} f(z)}$ is bounded. By Liouville's theorem, g is a constant, and hence so is f.

#3, p. 109. The set $\{1, 1/2, 1/3, \ldots\}$ is not discrete because it has a limit point at zero. So, if $f(1/n) = 0$ for all n and f is analytic, then $f(z) = 0$ for all z. It is easy to construct a function f that is analytic on $\mathbf{C} \setminus \{0\}$ and $f(1/n) = 0$ for all $n \geq 1$ though. For instance, consider $f(z) = e^{2\pi i/z}$.

#4, p. 109. Write the power-series expansion of sin:

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} \cdots = z + \sum_{n=1}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} = z + z^3 \sum_{k=0}^{\infty} \frac{(-1)^{k+1} z^{2k}}{(2k+3)!}$$

and deduce that $\sin z - z = z^3 g(z)$, where

$$g(z) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} z^{2k}}{(2k+3)!} = -\frac{1}{6} + \frac{z^2}{5!} - \frac{z^4}{7!} \cdots .$$

Note that g is analytic near zero, and nonzero. In fact, $g(0) = -1/6$. In particular, the order of zero of $\sin z - z$ is three.

#6, p. 109. Let $f(z) = \sin z$. Then, $f(2\pi n) = 0$ for all integers n. Let $U := D_r(0) \cup D_r(\{\pi\})$, and note that U is the union of two disjoint balls, and if r is sufficiently small, then f has exactly one zero in each ball. Suppose we could apply the first part of Theorem 3.4.2 to this example. Then, we could write $f(z) = z^k g(z)$, where g is analytic on U and nonvanishing on U. In particular, $f(\pi) = \pi^k g(\pi) \neq 0$, which is absurd.

#13, p. 109. Write

$$f(z) = \frac{1}{z} \cdot \frac{1}{1-z} \cdot \frac{1}{1+z}.$$

So we have simple poles respectively at 0, 1, and -1.

#1, p. 114. Compute directly first:

$$\left|z^2 - 1\right|^2 = (z^2 - 1)\,\overline{(z^2 - 1)} = |z|^4 - 2\mathrm{Re}(z^2) + 1.$$

Therefore,

$$\left|z^2 - 1\right|^2 = (z^2 - 1)\,\overline{(z^2 - 1)} = 2\left(1 - \mathrm{Re}(z^2)\right) \qquad \text{whenever } |z| = 1.$$

Write $z = x + iy$ to see that $z^2 = x^2 - y^2 + 2ixy$. Therefore, $\mathrm{Re}(z^2) = x^2 - y^2 = (\mathrm{Re}z)^2 - (\mathrm{Im}z)^2$, and therefore,

$$\left|z^2 - 1\right|^2 = 2\left(1 + (\mathrm{Im}z)^2 - (\mathrm{Re}z)^2\right) \qquad \text{whenever } |z| = 1.$$

On the other hand, if $|z| = 1$ the $(\mathrm{Re}z)^2 + (\mathrm{Im}z)^2 = 1$. Therefore,

$$\left|z^2 - 1\right|^2 = 4(\mathrm{Im}z)^2 \qquad \text{whenever } |z| = 1.$$

This is maximized precisely when $(\mathrm{Im}z)^2 = |z|^2 = 1$; that is when $\mathrm{Re}z = 0$; i.e., $z = \pm i$.

#3, p. 114. We wish to maximize $|z - 1|$ on the triangle $\triangle$ with vertices at 0, $1 + i$, and $1 - i$. The maxima occur at all the three endpoints of $\triangle$, and the value of the maximum is one.

#5, p. 115. Suppose, to the contrary, that $f(z) \neq 0$ for all $z \in D_1(0)$. Then $1/f$ is an analytic function on $D_1(0)$. And $|1/f(z)| = 1$ on the boundary of $D_1(0)$. By the maximum modulus principle, $|1/f(z)| \leq 1$ for all $z \in D_1(0)$. Equivalently, $|f(z)| \geq 1$ for all $z \in D_1(0)$.

On the other hand, $|f(z)| = 1$ on $D_1(0)$. Therefore by the maximum modulus principle, $|f(z)| \leq 1$ for all $z \in D_1(0)$. The preceding shows that $|f(z)| \geq 1$ and $|f(z)| \leq 1$—hence $|f(z)| = 1$—for all $z \in D_1(0)$. Another appeal to the maximum modulus principle shows that f is a constant in $D_1(0)$, which is a contradiction.