Solutions to Homework 4

Math 4200, Summer 2009

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#2, p. 71. $f(z) = (z^2 - 4)^{-1}$ is analytic inside the circle, and the circle surrounds a convex open set. Therefore, $\int_\gamma (z^2 - 4)^{-1} dz = 0$.

#6, p. 71. Choose and fix $\epsilon > 0$, and define

$$U := \{w \in \mathbf{C} : w \text{ is distance at most } \epsilon \text{ from the line from 1 to } z\}.$$

The line that goes from 1 to z is a strictly-positive distance δ away from the cutline of the principle branch of the logarithm. Therefore, if $\epsilon < \delta$, then U is an open and convex neighborhood of the line which does not intersect the cutline of log. Therefore, log is analytic on U and $(\log z)' = 1/z$ on U. In other words, the function $f(z) = 1/z$ is the antiderivative of $\log z$ on U. The result follows.

#12, p. 72. Let γ be a positively-oriented path that parametrizes the unit circle. Define $f(z) := e^z$ and $z_0 := 0$. By the Cauchy integral formula,

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0}\, dz.$$

The right-hand side is $(2\pi i)^{-1} \int_\gamma (e^z/z)\, dz$, whereas the left-hand side is $e^{z_0} = 1$. Therefore,

$$\int_\gamma \frac{e^z}{z}\, dz = 2\pi i.$$

#14, p. 72. The zeros of $z^2 - 1$ are $z = \pm 1$. Therefore, the function $f(z) := (z^2 - 1)^{-1}$ is analytic in the interior of the circle $\{w \in \mathbf{C} : |w - 1| = 1\}$ [though not on the boundary!]. By Cauchy's theorem,

$$\int_{|z|=1} \frac{dz}{z^2 - 1} = 0.$$

Also, f is analytic in the annulus $\{w : 1 < |w| < 3\}$. Therefore (why?),

$$\int_{|z|=3} \frac{dz}{z^2-1} - \int_{|z|=1} \frac{dz}{z^2-1} = 0.$$

Add the two displays to find that $\int_{|z|=3}(z^2-1)^{-1}dz = 0$.

#1, p. 87. Clearly,

$$\sup_{|z|\geq r}\left|\frac{1}{nz} - 0\right| = \sup_{|z|\geq r}\frac{1}{n|z|} = \frac{1}{nr} \to 0 \quad \text{as } n \to \infty.$$

Therefore, the sequence converges to zero uniformly on $\{z \in \mathbf{C} : \ |z| \geq r\}$, regardless of how small r is. On the other hand,

$$\sup_{z\neq 0}\left|\frac{1}{nz} - 0\right| = \sup_{z\neq 0}\frac{1}{n|z|} = \infty \not\to 0.$$

#6, p. 87. We have the bound $|k^2 - z| \geq k^2 - |z| \geq k^2 - r$, uniformly for all z such that $|z| \leq r$. Therefore, if $k^2 \geq 2r$ then

$$\left|\frac{1}{k^2-z}\right| \leq \frac{1}{k^2-r} \leq \frac{1}{k^2-(k^2/2)} = \frac{2}{k^2}.$$

That is,

$$\sum_{k\geq\sqrt{2r}}\left|\frac{1}{k^2-z}\right| \leq 2\sum_{k\geq\sqrt{2r}} k^{-2} < \infty. \tag{1}$$

Of course, $\sum_{1\leq k<\sqrt{2r}}(k^2-z)^{-1}$ converges uniformly for $z \in E_r$, because it is a finite sum. Therefore, the Weiertrass M-test tells us that the infinite sum is convergent uniformly on E_r.

#7, p. 87. If k is a positive integer and $z = x + iy \in \mathbf{C}$, then $k^z = k^x k^{iy} = k^x e^{iy\log k}$, and therefore $|k^z| = k^x = k^{\mathrm{Re}(z)}$. Therefore, if $\mathrm{Re}(z) > s$ with $s > 1$, then $|k^z| \geq k^s$, whence

$$\sum_{k=1}^{\infty}\left|\frac{1}{k^z}\right| \leq \sum_{k=1}^{\infty} k^{-s} < \infty.$$

The M-test does the rest.

#15, p. 88. Let $h(w) := (1+w)^{-1}$. Then, $h'(w) = -(1+w)^{-2}$, $h''(w) = 2(1+w)^{-3}$, etc. In particular, $h^{(n)}(0) = (-1)^n n!$, and so we have the power-series representation

$$\frac{1}{1+w} = \sum_{n=0}^{\infty} (-1)^n w^n.$$

The preceding power series converges uniformly on $\{z \in \mathbf{C} : |z| < 1\}$. Because we can integrate a power series term-by-term where it converges uniformly, it follows that if $|z| < 1$, then

$$\begin{aligned}\int_0^z \frac{dw}{1+w} &= \sum_{n=0}^{\infty} (-1)^n \int_0^z w^n \, dw = \sum_{n=0}^{\infty} (-1)^n \int_0^z \left(\frac{w^{n+1}}{n+1}\right)' dw \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{z^{n+1}}{n+1}.\end{aligned}$$

Now, this is a power series about zero whose nth coefficient is $c_n := (-1)^n/(n+1)$. Note that

$$|c_n|^{1/n} = (n+1)^{-1/n} = \exp\left\{-\frac{\log(n+1)}{n}\right\}.$$

Therefore, $\limsup_{n\to\infty} |c_n|^{1/n} = \lim_{n\to\infty} |c_n|^{1/n} = 1$. This is the radius of uniform convergence.

Therefore,

$$\sum_{n=0}^{\infty} (-1)^n \frac{z^{n+1}}{n+1} = \int_0^z \frac{dw}{1+w} = \ln(1+z),$$

for $|z| < 1$, because $1/(1+w)$ is the antiderivative of $\ln(1+w)$ in $D_1(0)$.

#9, p. 95. We did this as a theorem in the lectures (see also Theorem 3.4.2).

#11, p. 95. Correction: $p(z) = a_3 z^3 + a_2 z^2 + a_1 z + a_0$. Now suppose $|p(z)| \leq 1$ on $\{z \in \mathbf{C} : |z| = 1\}$. According to Cauchy's estimate (Theorem 3.2.9), $6|a_3| = |p'''(0)| \leq 3! = 6$. Therefore, $|a_3| \leq 1$ (and hence ≤ 6).