Solutions to Homework 3

Math 4200, Summer 2009

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#3, p. 50. Well ... the answer is $2\pi e^{i(n+m)}$. But I think the question is meant to ask you to find $\int_0^{2\pi} e^{int}e^{imt}\,dt$. This is the same as asking for $\int_0^{2\pi} e^{iqt}\,dt$ where $q = n+m$ is an arbitrary integer. We compute this the only way we know how:

$$\int_0^{2\pi} e^{iqt}\,dt = \int_0^{2\pi} \cos(qt)\,dt + i\int_0^{2\pi} \sin(qt)\,dt.$$

Because q is an integer, the sine integral is always zero. So is the cosine integral unless $q = 0$; in that case the integral is 2π. Therefore,

$$\int_0^{2\pi} e^{int}e^{imt}\,dt = \begin{cases} 2\pi & \text{if } n = -m, \\ 0 & \text{otherwise.} \end{cases}$$

#10, p. 50. Because $\gamma'(t) = 3ie^{it}$,

$$\int_\gamma \frac{1}{z}\,dz = \frac{1}{3}\int_0^{2\pi} e^{-it} \cdot 3ie^{it}\,dt = 2\pi i.$$

[This says that this circle goes around the origin once.] On the other hand,

$$\int_\gamma \bar{z}\,dz = \int_0^{2\pi} 3e^{-it} \cdot 3ie^{it}\,dt = 18\pi i.$$

#13, p. 50. No: For instance, consider problem #10 that we solved earlier: γ is the unit circle that goes around the origin once; and $f(z) := \bar{z}$. Then,

$$\mathrm{Re}\left(\int_\gamma \bar{z}\,dz\right) = \mathrm{Re}(18\pi i) = 0.$$

On the other hand,

$$\int_\gamma \mathrm{Re}(\bar{z})\,dz = \int_0^{2\pi} \mathrm{Re}(e^{-it})3ie^{it}\,dt = 3i\int_0^{2\pi} \cos(t)e^{it}\,dt.$$

It is easy to compute this integral:

$$\int_0^{2\pi} \cos(t)e^{it}\,dt = \int_0^{2\pi} (\cos t)^2\,dt + i\int_0^{2\pi} \cos t \sin t\,dt = \pi.$$

Therefore,

$$\int_\gamma \mathrm{Re}(\bar{z})\,dz = 3\pi i \neq 0 = \mathrm{Re}\left(\int_\gamma \bar{z}\,dz\right).$$

#2, p. 57. $\gamma(t) = e^{it}$ for $0 \le t \le 4\pi$ works, and $\gamma'(t) = ie^{it}$. Therefore,

$$\int_\gamma \frac{1}{z}\,dz = \int_0^{4\pi} \frac{1}{e^{it}} \cdot ie^{it}\,dt = 4\pi i.$$

#6, p. 57. Think simple; for instance, $\alpha(t) = 2 + 3t$ for $0 \le t \le 1$.

#10, p. 58. A polynomial p is of the form $p(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n$ for $c_1, \ldots, c_n \in \mathbf{C}$. Note that

$$\int_\gamma p(z)\,dz = \sum_{j=0}^{n} c_j \int_\gamma z^j\,dz.$$

Therefore, it suffices to check that $\int_\gamma z^j\,dz = 0$ for all $j \ge 0$. But $z^j = f'(z)$, where $f(z) = (j+1)^{-1} z^{j+1}$. Therefore, for *any* path $\gamma : [a\,, b] \to \mathbf{C}$,

$$\int_\gamma z^j\,dz = \frac{(\gamma(b))^{j+1} - (\gamma(a))^{j+1}}{j+1}.$$

And this is zero if the path is closed.

Alternatively, we can just compute $\int_\gamma z^j\,dz$ and learn this the direct way:

$$\int_\gamma z^j\,dz = \int_0^{2\pi} (\gamma(t))^j \gamma'(t)\,dt = \int_0^{2\pi} e^{itj} ie^{it}\,dt = i\int_0^{2\pi} e^{2it(j+1)}\,dt = 0;$$

see my addition to problem #30 on page 50 for the last calculation.

#14, p. 65. It might help to draw the pictures as I describe them.

Let Δ be a triangle with one horizontal side and the remaining corner sitting above that side; so your Δ should be pointing up when you have drawn it. First, let us consider the case that I is the bottom side of I.

Because f is continuous on Δ and the triangle is bounded and closed, it follows that f is uniformly continuous on and in the triangle Δ.This means that for all $\epsilon > 0$, there exists $\delta > 0$ such that $|f(z) - f(w)| < \epsilon$ whenever w and z are two points in Δ with $|z - w| < \delta$. Choose and fix ϵ, and therefore also δ.

Now draw a line I' direct above and parallel to I, such that the distance between I and I' is $1/n$, where n is much larger than the diameter of Δ [we will choose n shortly]. Let L denote the length of I'. Start from the bottom-left point where I' meets the left-side of Δ and draw vertical line segments that join I' to I such that the distance between any two of these vertical line segments is exactly L/n. In this way, you will end up with exactly $n + 1$ line segments (check!).

What you have before you is a partition of the region in Δ that is enclosed between I and I'; and you have partitioned that region into $n + 2$ parts: n are squares of length L/n; and 2 are right triangles. Each triangle has height $1/n$ and each base of each triangle has length constant times $1/n$ [the constant is the tangent of the angle made between that side of Δ and the horizontal line, so the two constants could be different].

Note that the perimeter length of every one of the n bottom squares is $4/n$, and for the two bottom triangles it is c_0/n and c_{n+1}/n; c_0 for the left-most and c_{n+1} for the right-most [say]. Note that each of the two constants c_0 and c_{n+1} is the tangent of a bottom-corner angle of Δ and does not depend on n.

Define

$$\gamma := \max(4\,, c_0\,, c_{n+1}); \tag{1}$$

γ does not depend on n. γ/n is the largest perimeter length among the $n + 2$ little shapes at the bottom of Δ.

Now we choose n so large that $\gamma/n < \delta$. Also choose $n + 2$ points $w_0, w_1, \ldots, w_n, w_{n+1}$ such that each w_j lies in the center of the jth bottom square for $1 \leq j \leq n$; w_0 lies in the center of the bottom-left triangle; and w_{n+1} lies in the center of the bottom-right triangle.

Consider any one of the bottom squares and/or triangles. Because of the way we chose n, any point z in that shape is within δ of its center. For instance, in the jth bottom square, $|z - w_j| < \delta$ for all z in that square. This show that $|f(z) - f(w_j)| < \epsilon$, uniformly for all bottom-squares [and bottom-triangles]. Let Q_j denote the jth one of those shapes ($0 \le j \le n+1$), starting from the left-most. Then, the center of each Q_j is w_j, and

$$\left| \int_{\partial Q_j} f(z)\, dz - \int_{\partial Q_j} f(w_j)\, dz \right| < \epsilon \ell(Q_j) \le \frac{\epsilon\gamma}{n}.$$

Since $f(w_j)$ is a constant [an antiderivative!], $\int_{\partial Q_j} f(w_j)\, dz = 0$. Therefore, if S denotes the bottom strip between I' and I, then

$$\left| \int_{\partial S} f(z)\, dz \right| \le \sum_{j=0}^{n+1} \left| \int_{\partial Q_j} f(z)\, dz \right| \le \epsilon\gamma \frac{n+1}{n}.$$

This is valid for all integers n such that $\gamma/n < \delta$. Let $n \to \infty$ and then $\epsilon \to 0$, in this order, to deduce that $\int_{\partial S} f(z)\, dz = 0$.

Note that Δ is comprised of a triangle Δ' and S; Δ' is sitting atop S. Therefore, $\int_{\partial\Delta} f(z)\, dz = \int_{\partial\Delta'} f(z)\, dz + \int_{\partial S} f(z)\, dz$. We just proved that $\int_{\partial S} f(z)\, dz = 0$. Cauchy's theorem of your textbook shows that $\int_{\partial\Delta'} f(x)\, dz = 0$. Therefore, $\int_{\partial\Delta} f(z)\, dz = 0$, as desired.

Now if I is *any* horizontal interval in the interior [or on the lower boundary] of Δ, then draw a triangle Δ', pointing up, such that Δ' lies entirely in Δ and its base is I. Triangulate the rest of Δ, and use the preceding part, together with Cauchy's theorem from your text to conclude that $\int_{\partial\Delta} f(z)\, dz = 0$. If I is not horizontal, or Δ does not have a horizontal base, then simple alterations to the picture suffice. I will leave them to you.

#15, p. 65. $\int_\gamma f(z)\, dz = 0$, as long as the boundary of γ can be described by continuously joining a finite number of triangles.