Solutions to Homework 2

Math 4200, Summer 2009

June 11, 2009

#4, p. 20. If $e^z = 1 + i\sqrt{3}$, then $z = \log(1 + i\sqrt{3}) + 2\pi ki$, for some integer $k = 0, \pm 1, \pm 2, \ldots$. It suffices to compute $\log(1 + i\sqrt{3})$. But that is

$$\log\left(1 + i\sqrt{3}\right) = \log\left|1 + i\sqrt{3}\right| + i\arg\left(1 + i\sqrt{3}\right) = \log 2 + \frac{\pi}{3}i.$$

#9, p. 20. Because $e^z = e^{\operatorname{Re} z} \cdot e^{i \operatorname{Im} z} = e^z(\cos \operatorname{Re} z + \sin \operatorname{Im} z)$, it follows that $e^z \in \mathbf{R}$ if and only if $\sin \operatorname{Im} z = 0$. That is, $\operatorname{Im} z = 0, \pm\pi, \pm 2\pi, \ldots$ [plot these point!]. Similarly, e^z is purely imaginary if and only if $\operatorname{Im} z = \pm\pi/2, \pm 3\pi/2, \ldots$ [plot these also!].

#4, 27. In polar coordinates, $1 - i = \sqrt{2}e^{-i\pi/4}$ and $1 + i = \sqrt{2}e^{i\pi/4}$. Therefore,

$$\frac{(1-i)^7}{1+i} = \frac{2^{7/2}e^{-i7\pi/4}}{2^{1/2}e^{i\pi/4}} = 2^3 e^{-2\pi i} = 8.$$

#10, p. 27. These are simple computations: (a) $-\pi/2$; (b) $3\pi/2$; (c) 2π.

#14, p. 37. (a) Let $A := \{w \in U : |w| < r\}$. Then A is open in U [in fact, A is the open ball of radius e inside U]. It suffices to verify the following: **Claim.** $f^{-1}(A) := \{z \in U : |f(z)| < r\}$.

This is straight-forward. But I will prove it once so that you can remember how this sort of thing is proved. For the sake of convenience define $\tilde{A} := \{z \in U : |f(z)| < r\}$. We wish to show that $f^{-1}(A) = \tilde{A}$ by first establishing that $f^{-1}(A) \subseteq \tilde{A}$, and then that $\tilde{A} \subseteq f^{-1}(A)$.

1. If $w \in f^{-1}(A)$, then $f(w) \in A$, and therefore $|f(w)| < r$ by the definition of A. This shows that $f^{-1}(A) \subseteq \tilde{A}$. **2.** If $z \in \tilde{A}$, then $|f(z)| < r$, whence $f(z) \in A$. That is, $z \in f^{-1}(A)$. This proves the remaining portion that $\tilde{A} \subseteq f^{-1}(A)$.

(b) This part is similar to (a), but now in place of A, we consider $A_1 := \{w \in U : \operatorname{Re} w < r\}$. If we showed that A_1 is open then it follows from the method of (a) that $f^{-1}(A_1) = \{z \in U : \operatorname{Re} f(z) < r\}$ is also open. Define $g(z) := \operatorname{Re} z$. Then, g is continuous and $A_1 = g^{-1}(A)$, which is open by part (a).

#7, p. 43. Apply complex chain rule [Theorem 2.2.7, p. 39] to $f(z) = g(h(z))$, where $g(z) := \exp(z)$ and $h(z) := z^3$ to find that $f'(z) = 3z^2 \exp(z^3)$.

#11, p. 44. Suppose f is analytic on $\mathbf{C}$, and real-valued. Write $f = u + iv$, to see that $v \equiv 0$. By the Cauchy–Riemann, equations,

$$f'(z) = u_x(z) = -iu_y(z).$$

Because u_x and u_y are real-valued, the preceding tells us that they are both zero [zero being the only number in $\mathbf{C}$ that is both real and purely imaginary]. Therefore, u does not depend on x and y. That is, f is a constant.

12, p. 44. By chain rule, $u_r = u_x \cdot \frac{\partial x}{\partial r} + u_y \cdot \frac{\partial y}{\partial r} = u_x \cdot \cos\theta + u_y \cdot \sin\theta$. Recall that the Cauchy–Riemann equations tell us that $u_x = v_y$ and $u_y = -v_x$. Apply these in the previous display to find that

$$u_r = v_y \cos\theta - v_x \sin\theta. \tag{1}$$

Another round of chain rule tells that

$$u_\theta = -r\left(v_x \cos\theta + v_y \sin\theta\right). \tag{2}$$

Also,

$$v_r = v_x \cos\theta + v_y \sin\theta, \tag{3}$$

and

$$v_\theta = -r \sin\theta + rv_y \cos\theta. \tag{4}$$

Compare (4) to (1) to find that $v_\theta = ru_r$; this is one of the desired equations. For the other, compare (3) to (2) to find that $u_\theta = -rv_r$.