Math 3210–1, Summer 2016
Solutions to Assignment 1

1.1. #2. We are asked to show that the following two propositions are both true:

(a) $x \in A \cap (B \cup C) \Rightarrow x \in (A \cap B) \cup (A \cap C)$; and

(b) $x \in (A \cap B) \cup (A \cap C) \Rightarrow x \in A \cap (B \cup C)$.

To prove (a) let $x \in A \cap (B \cup C)$. Then, $x \in A$ and $x \in B \cup C$ (by the definition of "$\cap$"). Also, saying that "$x \in B \cup C$" is the same as saying that "either $x \in B$ or $x \in C$." In the first case we deduce that $x \in A$ and $x \in B$; in the second that $x \in A$ and $x \in C$. In other words, $x \in (A \cap B) \cup (A \cap C)$, as desired.

To prove (b) let $x \in (A \cap B) \cup (A \cap C)$ and observe that either $x \in A \cap B$ [that is, $x \in A$ and $x \in B$] or $x \in A \cap C$. In either case, $x \in A$ and either $x \in B$ or $x \in C$. That is, $x \in A \cap (B \cup C)$.

1.1. #4. We claim that the answer is $[0\,,1]$. That is, we claim that

$$\bigcap_{\substack{a<0\\ b>1}} (a\,,b) = [0\,,1].$$

If $x \in [0\,,1]$ then $0 \le x \le 1$ whence $a < x < b$ for every $a < 0$ and $b > 1$. It follows that $[0\,,1] \subset (a\,,b)$ for all $a < 0$ and $b > 1$, whence

$$[0\,,1] \subset \bigcap_{\substack{a<0\\ b>1}} (a\,,b).$$

In order to complete the proof, we establish the converse inclusion. By contraposition, we plan to verify that

$$x \notin [0\,,1] \Rightarrow x \notin \bigcap_{\substack{a<0\\ b>1}} (a\,,b). \tag{1}$$

If $x \notin [0\,,1]$ then either $x < 0$ or $x > 1$. On one hand, if $x < 0$, then we can find an open interval $(a\,,b)$ that includes $[0\,,1]$ but does not contain x. For example, a could be $x - 1$ and b could be 2. On the other hand, if $x > 1$ then we can still find an open interval $(a\,,b)$ that includes $[0\,,1]$ but does not contain x. For example, a could be -1 and b could be $x + 1$. In either case, we see that if $x \notin [0\,,1]$ then x is not in $(a\,,b)$ for some $a < 0$ and $b > 1$. This proves (1).

1.1. #5. We claim that

$$\bigcap_{\substack{a\le 0\\ b\ge 1}}[a\,,b] = (0\,,1).$$

If $x\in(0\,,1)$, then $x\in[a\,,b]$ for all $a\le 0$ and $b\ge 1$, and hence

$$\bigcap_{\substack{a\le 0\\ b\ge 1}}[a\,,b] \supset (0\,,1).$$

On the other hand, if $x\notin(0\,,1)$, then $x\notin[a\,,b]$ for some $a\le 0$ and $b\ge 0$ for the following reason: If $x\le 0$, then $x\notin[x/2\,,1]$; and if $x\ge 1$ then $x\notin[0\,,(x+1)/2]$.

1.1. #13. Let $f:\mathbb{R}\to\mathbb{R}$ be $f(x):=x^2$ for all $x\in\mathbb{R}$. Set $E=[-1\,,1]$ and $F=[0\,,1]$. Then, $f(E)=f(F)=[0\,,1]$—in particular, $f(E)\setminus f(F)=\varnothing$—and $f(E\setminus F)=f([-1\,,0])=[0\,,1]\neq\varnothing$.

1.2. #2. First consider the case that $n=1$. In that case, we are tasked to prove that $m+1\neq 1$ for all $m\in\mathbb{N}$. But this is a part of the construction of the natural numbers.

Next (induction hypothesis) assume that $m+n\neq n$ for some $n\in\mathbb{N}$ and all $m\in\mathbb{N}$. We need to prove that $m+(n+1)\neq n+1$. Subtract one from both sides to see that our goal is to prove that $m+n\neq n$, which is a consequence of the induction hypothesis.

1.2. #14. First of all, $x_1=1$, $x_2=\frac{1}{2}$, and $x_3=\frac{2}{3}$; therefore, x_3 falls between x_1 and x_2. This is the base case. Now set up the induction hypothesis: Suppose x_{n+2} lies between x_n and x_{n+1}. That is,

$$x_n < x_{n+2} < x_{n+1} \quad\text{or}\quad x_{n+1} < x_{n+2} < x_n. \tag{2}$$

If $x_n < x_{n+2} < x_{n+2}$, then

$$x_{n+3} = \frac{1}{1+x_{n+2}} < \frac{1}{x_n} = x_{n+1} \quad\text{and}\quad x_{n+3} = \frac{1}{1+x_{n+2}} > \frac{1}{x_{n+1}} = x_{n+2}.$$

If $x_{n+1} < x_{n+2} < x_n$, then

$$x_{n+3} = \frac{1}{1+x_{n+2}} > \frac{1}{x_n} = x_{n+1} \quad\text{and}\quad x_{n+3} = \frac{1}{1+x_{n+2}} < \frac{1}{x_{n+1}} = x_{n+2}.$$

In any case, it follows that if (2) held for some $n\in\mathbb{N}$ then so would

$$x_{n+1} < x_{n+3} < x_{n+2} \quad\text{or}\quad x_{n+2} < x_{n+3} < x_{n+1}.$$

1.2. #17. We simply write it out:

$$\begin{aligned}\binom{n}{k-1}+\binom{n}{k} &= \frac{n!}{(k-1)!(n-k+1)!}+\frac{n!}{k!(n-k)!}\\ &= \frac{k\cdot n!}{k!(n-k+1)\cdot(n-k)!}+\frac{n!}{k!(n-k)!} = \frac{n!}{k!(n-k)!}\left[\frac{k}{n-k+1}+1\right]\\ &= \frac{n!}{k!(n-k)!}\cdot\frac{n+1}{n-k+1}\\ &= \binom{n+1}{k}.\end{aligned}$$

1.3. #8. Recall that "$a > 0$" means that "$a \geq 0$ and $a \neq 0$."

Now suppose $x > 0$ and $y > 0$. If x and y are rationals, then this is easy: Write $x = n/m$ and $y = k/l$ for $n, m, k, l \in \mathbb{N}$, and hence $xy = (nk)/(ml)$ is the ratio of two natural numbers whence > 0. In the general case we can find two rationals p and q such that $x \geq p > 0$ and $y \geq q > 0$. Thus, $xy \geq pq > 0$.

1.3. #9. If $x > 0$ then $x \neq 0$ and so x^{-1} is a well-defined real number [by the construction of $\mathbb{R}$ in this chapter; be sure that you understand that this is not an intuitive statement!]. We are asked to prove that $x^{-1} \neq 0$ and $x^{-1} \geq 0$. Note that $(x^{-1})^{-1} = x$ because $xx^{-1} = 1$ and the reciprocal of a is defined uniquely for all nonzero $a \in \mathbb{R}$. In particular, $x^{-1} \neq 0$ because x^{-1} has a reciprocal. If it were the case that $x^{-1} < 0$, then $1 = xx^{-1}$ would be the product of a positive and a negative number, whence negative. But $1 > 0$.

1.3. #10. Suppose to the contrary that $y^{-1} \geq x^{-1}$. Then, $1 = yy^{-1} \geq yx^{-1}$. Multiply both sides by $x > 0$ to see that $x \geq y$, which is a contradiction.

1.3. #11. By Theorem 1.3.9, if $x^2 = 5$ had a rational positive solution then that solution would have to be an integer. In other words, if $\sqrt{5}$ were rational then $\sqrt{5}$ would have to be an integer. But $2 = \sqrt{4} < \sqrt{5} < \sqrt{9} = 3$, and there are no integers that lie strictly between 2 and 3.