Math 3210-1, Midterm 3

Thursday July 14, 2016

Name: ______________________________

This exams is comprised of 4 questions, on 4 pages, and for a total of 50 points.

1. (10 points) Define $f(x) = \sqrt{x}$ for all $x \in (0\,,1)$. Is f uniformly continuous?

Solution. Yes. Here is one way to reason this fact: If $0 < x, y \leq 1$ then

$$x - y = \left(\sqrt{x} - \sqrt{y}\right)\left(\sqrt{x} + \sqrt{y}\right).$$

Therefore,

$$\sqrt{x} - \sqrt{y} = \frac{x - y}{\sqrt{x} + \sqrt{y}}.$$

The same identity holds if one—but not both—of x and y are zero. Therefore, for every $x \in [0\,,1]$, we have $\lim_{y \to x} \sqrt{y} = \sqrt{x}$. In other words, $f(x) = \sqrt{x}$—viewed as a function on the closed and bounded interval $[0\,,1]$—is continuous. In particular, $\sqrt{x}$—viewed as a function on the open interval $(0\,,1)$—has a continuous extension to $[0\,,1]$ and is therefore uniformly continuous on $(0\,,1)$ by one of the theorems that we proved earlier in the lectures.

2. (10 points) Let $f : [0\,,1] \to \mathbb{R}$ be a function that satisfies

$$f(0) = 5 \qquad \text{and} \qquad |f(x) - f(y)| \le |x - y|^2 \qquad \forall x, y \in [0\,,1].$$

Compute $f(x)$ for every $x \in [0\,,1]$. (Hint: What is f'?)

Solution. For every $x \in (0\,,1)$,

$$\left|\frac{f(x+h) - f(x)}{h}\right| \le h \to 0 \qquad \text{as } h \to 0.$$

Therefore, f is differentiable on $(0\,,1)$ and $f'(x) = 0$ for all $x \in (0\,,1)$. By the mean-value theorem, f is a constant on $(0\,,1)$. On the other hand, f is continuous on $[0\,,1]$. Therefore, f is a constant on $[0\,,1]$. Since $f(0) = 5$ it follows that $f(x) = 5$ for all $x \in [0\,,1]$.

3. (10 points) Prove that $|\cos^2(x) - \cos^2(y)| \leq 2|x - y|$ for all $x, y \in \mathbb{R}$.

Solution. $|\cos^2(x) - \cos^2(y)| = |\cos x - \cos y||\cos x + \cos y|$. By the triangle inequality, $|\cos x + \cos y| \leq |\cos x| + |\cos y| \leq 2$. Therefore, it remains to prove that $|\cos x - \cos y| \leq |x - y|$ for all $x, y \in \mathbb{R}$.

Without loss of generality, suppose $x < y$. Then, the mean-value theorem ensures that there exists $c \in (x\,, y)$ such that

$$\frac{\cos y - \cos x}{y - x} = -\sin c.$$

Since $|-\sin c| \leq 1$, this proves that $|\cos x - \cos y| \leq |x - y|$ and concludes the proof.

4. (20 points total) Let $\alpha > 0$ be fixed.

   (a) (10 points) Prove that if $\alpha \le 1$, then
$$(1+x)^\alpha \le 1 + \alpha x \qquad \forall x \ge 0.$$

   (b) (10 points) Prove that if $\alpha > 1$, then
$$(1+x)^\alpha > 1 + \alpha x \qquad \forall x \ge 0.$$

Solution. There are certain, special, values of α for which the problem is easy to verify directly. For instance if $\alpha = 1$ then there is nothing to prove. Or, for that matter, when $\alpha = 2$ we have
$$(1+x)^\alpha = (1+x)^2 = 1 + 2x + x^2 \ge 1 + 2x \qquad \forall x \ge 0.$$

We are asked to study the problem for general α, however.

Define
$$f(x) = (1+x)^\alpha - \alpha x \qquad \forall x \ge 0.$$
Then f is differentiable on $(0\,,\infty)$ and continuous on $[0\,,1]$; moreover,
$$f'(x) = \alpha(1+x)^{\alpha-1} - \alpha \qquad \forall x > 0.$$

If $\alpha \in (0\,,1]$, then $(1+x)^{\alpha-1} \le 1$, whence $f'(x) \le 0$ for all $x > 0$. In particular, $f(x) \le f(0) = 1$ for all x. This proves (a).

Conversely, if $\alpha > 1$ then $(1+x)^{\alpha-1} > 1$, whence $f'(x) > 0$ for all $x > 0$. In particular, $f(x) \ge f(0) = 1$ for all x. This proves (b).