Math 3210-1, Midterm 2

Thursday June 23, 2016

Name: ________________________________

This exams is comprised of 5 questions, on 5 pages, and for a total of 30 points.

1. (10 points total; 5 points each) Suppose that $a_1, a_2, \ldots$ is a sequence of real numbers that satisfy $a_n < 1$ for all $n \in \mathbb{N}$.

   (a) Prove that $\limsup_{n\to\infty} a_n \le 1$.

   (b) Can one improve the preceding to $\limsup_{n\to\infty} a_n < 1$? Justify your answer.

   Solution. (a) Because 1 is an upper bound for $\{a_n\}_{n=1}^{\infty}$, 1 is greater than or equal to the smallest upper bound for $\{a_n\}_{n=1}^{\infty}$. In other words, $\sup_{n\ge1} a_n \le 1$. Therefore in particular, $s_k = \sup_{n\ge k} a_n \le 1$ for all $k \ge 1$. Let $k \to \infty$ to see that $\limsup_{n\to\infty} a_n = \lim_{k\to\infty} s_k \le 1$.

   (b) No. For instance, consider the sequence $a_n = 1 - (1/n)$ for all $n \ge 1$.

2. (5 points) We know from the Bolzano–Weierstrass theorem that every bounded sequence has a convergent subsequence. Is the converse true? In other words, is it true that every sequence that has a convergent subsequence must be bounded? Justify your answer.

   Solution. No. For example consider the sequence

$$a_n = \begin{cases} n & \text{if } n \text{ is even,} \\ 1/n & \text{if } n \text{ is odd.} \end{cases}$$

   Then, $\{a_n\}_{n=1}^{\infty}$ is unbounded, but $a_{2k+1} \to 0$ as $k \to \infty$.

3. (5 points) Define

$$f(x) = 1 + x^2, \qquad g(x) = \frac{1}{1+x^2} \qquad \forall x \in [{}^1\!/\!{}_2\,, 1].$$

If it is possible, then compute $(f \circ g)(1/2)$ and $(g \circ f)(1/2)$. If it is not possible explain clearly why it is not possible to do that.

Solution. Because g is decreasing, $g(1/2) = {}^4\!/\!{}_5$, and $g(1) = {}^1\!/\!{}_2$, it follows that $g(x) \in [{}^1\!/\!{}_2\,, 1]$ for all $x \in [{}^1\!/\!{}_2\,, 1]$. Therefore, $f \circ g : [0\,, {}^1\!/\!{}_2\,, 1] \to \mathbb{R}$ is well defined and

$$(f \circ g)(1/2) = f(g(1/2)) = 1 + |g(1/2)|^2 = 1 + |{}^4\!/\!{}_5|^2 = 1 + \frac{16}{25} = \frac{41}{25}.$$

On the other hand, $(g \circ f)(1/2)$ is not well defined because $f(1/2) = 5/4 > 1$.

4. (5 points) Suppose $f : [0\,,1] \to \mathbb{R}$ is a continuous function that satisfies $f(x) \le 2$ for all $x \in [0\,,1]$ except possibly $x = 1/2$. Prove that $f(1/2) \le 2$.

Solution. For every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $y \in [0\,,1]$,

$$|y - \tfrac{1}{2}| < \delta \quad \Rightarrow \quad |f(1/2) - f(y)| < \varepsilon.$$

In particular, for every $y \in [0\,,1]$,

$$\tfrac{1}{2} - \delta < y < \tfrac{1}{2} + \delta \quad \Rightarrow \quad f(1/2) \le f(y) + \varepsilon.$$

If in addition $y \neq 1/2$ then $f(y) \le 2$, and hence $f(1/2) \le 2 + \varepsilon\ \forall \varepsilon > 0$. This implies that $f(1/2) \le 2$.

5. (5 points) True or False: Every strictly increasing function on $[0\,,1]$ is continuous. Justify your answer.

Solution. False. For example, consider

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 + x & \text{if } x \in (0\,,1]. \end{cases}$$

Claim 1. f is strictly increasing.
Claim 2. f is not continuous at 0.
Proof of Claim 1. If $0 \le x < y \le 1$, then $f(y) = 1 + y > 1 + x \ge f(x)$.
Proof of Claim 2. f is not continuous at zero; for instance set $\varepsilon = 1/2$. Then for all $\delta > 0$ and $y \in [0\,,1]$,

$$|y| < \delta \Rightarrow |f(y) - f(0)| = 1 + y \ge 1 > \varepsilon.$$